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**BULLETIN OF THE
AMERICAN
MATHEMATICAL SOCIETY**

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

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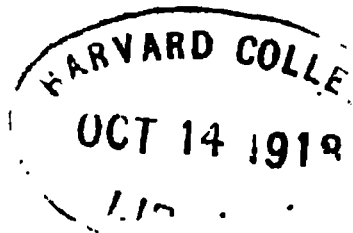
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PART I
FUNCTIONALS AND THEIR APPLICATIONS
SELECTED TOPICS, INCLUDING
INTEGRAL EQUATIONS

BY
GRIFFITH CONRAD EVANS

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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

A CONSPECTUS OF THE MODERN THEORY OF DIVERGENT SERIES.*

BY PROFESSOR WALTER B. FORD.

It will doubtless be recalled by all here present that the first volume of the encyclopedia of mathematics, as originally published in German, was issued in hefts, or parts, each prepared by a special collaborator well qualified for his task. The completion of the volume occupied a period of about six years and resulted in a finished work of no less than 1,128 pages. This was followed some years later by the second volume, which was still larger, containing in all its parts 1,154 pages. The two volumes taken together were intended to cover the entire range of pure mathematical analysis (as distinguished from geometry, mechanics, and other applications), each important branch being treated in its essentials and the amount of space allotted to any one topic being proportionate, at least roughly, to its relative importance. Out of the grand total of 2,282 pages thus presented it may now be remarked that a little less than 7 pages were devoted to that particular topic specified as "divergent series." Thus we have a ratio of 7 to 2,282, or about three tenths of one per cent, which may fairly be taken as the measure of interest in this topic at the time when the encyclopedia began to appear; that is, in the neighborhood of twenty years ago. The 7 pages in question, as we come to examine them, seem directed mainly to showing by means of simple illustrations that the processes by which Euler arrived at certain noteworthy results while

* Address of the retiring chairman of the Chicago Section of the American Mathematical Society, read at the joint meeting of the Section and of the Mathematical Association of America at Chicago, December 28, 1917.

dealing with divergent series are in themselves altogether unjustified and unscientific, the correctness of his results being in the nature of a happy circumstance arising out of the inherent character of the particular series he happened to be dealing with. On the whole, the short article conveys a rather gloomy outlook for this entire field of study, especially as regards any attempts that may be made to put it upon a truly scientific basis. Without digressing further upon the attitude of the encyclopedia, I take for granted that to-day we are all quite willing to agree that, contrary to any predictions that may have been made in the past, divergent series have now come to occupy a prominent place in analysis and one that bids fair to be permanent. How, it may be asked, has this happened? What has been done during these twenty years that really constitutes a vital advance in this field of study? Can we say that divergent series are at last upon a scientific basis? These are fair questions and it is to them that I would respectfully direct your attention for a few moments this afternoon. In answering, I shall attempt no more than an outline or conspectus of the situation as I myself have come to regard it and in this I am conscious beforehand that my own feelings, at least at some points, may not be universally acceptable; yet I shall venture all with equal candor. I shall not attempt to detail a large body of more or less intricate theorems and results, but I shall endeavor in a general way to show what seems central to me both as to the logical position of the present day theory and its applications.

In the first place, if I may refer again to Euler, we should of course recognize clearly at the outset that in his day the class of series which we now call divergent had not been clearly separated off by itself and in this sense it really had no well-defined meaning. It is true that certain individuals of the class had been studied more or less extensively and in ways which we are now bound to regard as interesting because of their curiosity, as for example the assignment of the sum $\frac{1}{2}$ to the oscillating divergent series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots,$$

but divergent series as a class had not been carefully defined. Not until the time of Cauchy and Abel did they take on an exact sense and it was then through a purely negative process.

In fact, they were then defined, as they still are to-day, as being all those series which do *not* satisfy the particular definition for convergent series which was formulated by these two great mathematicians. On this plan, convergence means nothing more or less than that the sum of the first n terms approaches a limit as n increases indefinitely, while divergence means simply that this particular limit does not exist. However natural this sort of a classification of series may seem to us, owing to our having been born and brought up with it, we must recognize that it involves after all a large element of arbitrariness. In fact, the only reason why the fundamental distinction between series should be made to hinge upon the existence of this particular limit lies in the fact that those series for which it exists are an admittedly important class. Of course there is no adequate reason in this to account for the way in which the excluded or divergent class were so long held in utter disrepute by the successors of Abel and Cauchy and continue still to repel us, except it be the psychological fact that whatever is excluded from a certain good class we instinctively think of as forming a bad class. However, to banish a whole class of series from analysis simply because it does not bear the stamp of a certain branded variety may easily be and is too hasty and rough a handling of their case. It makes no allowance for the different degrees of respectability which, though divergent, they may still possess. Thus it is, in substance, that the modern studies on divergent series may be said to have arisen. They are a reaction against the underlying arbitrariness of the Abel-Cauchy distinction and the ruthless and unjustified exclusions which this particular distinction has tended to produce. If we inquire just what the avenue of approach has been to this subject, it may be said that the first tangible result was the creation of the so-called "sum formulas," the essential feature of any such formula being that it shall not only serve, in case it can be evaluated, to give us the sum of any convergent series, but it shall continue to preserve a meaning when applied in the same way to certain divergent series, thus associating, or assigning, definite numerical values or sums to them also. The spirit of this procedure is, of course, nothing more or less than that common to all extensions of idea in mathematics. Just as in the theory of functions of a complex variable the sine of x , or any other function such as there considered, comes to have a

meaning for complex values of x only by virtue of a certain definitional formula so constructed as to give results when x is real that are in accord with the known properties of the sine in the real domain, so in the modern studies on divergent series sums are assigned to such series only by virtue of sum formulas so constructed as to yield results when applied to convergent series that are in accord with the known facts regarding such series. I have purposely mentioned this particular illustration because by carrying it one step farther, as I shall now do, I shall be able to bring out another feature of the present day situation as regards divergent series, and this time we shall discover that the theory is by no means in a perfectly satisfactory logical state as yet, being in this respect quite different from any well established body of doctrine such as the theory of functions of a complex variable. In fact, it will be recalled that it is a vital feature of the complex variable theory that no two different definitions for any one given function, as $\sin x$, are possible. In other words, it is demonstrable that if any two definitions for the function agree throughout the real domain, they will necessarily agree also throughout the complex domain. This property of uniqueness, if we may so designate it, is brought about by imposing a fundamental limitation at the very outset of the theory, namely, that only those functions shall be retained which belong to the class known as monogenic, or analytic. Thus, the logical coherence of the complex variable theory is bought at the price of a far reaching initial restriction, yet one which, as we know, is not so serious but what we are left with a class of functions of great interest in themselves and in their applications. In the theory of divergent series, on the other hand, as it exists today we are in possession of a large number of sum formulas each applicable in the sense before described to a divergent series and all agreeing with each other so far as convergent series are concerned, but *not* agreeing in general in the sums they prescribe to a given divergent series. In a word, the situation is analogous to that which the theory of functions would present in case the fundamental limitation as to the character of the functions considered were to be removed. How, you will at once inquire, can there be any general theory worthy of the name on such a loose plan as this? Strictly speaking, there cannot. However, all that is lacking in order that we have a bona fide theory is that we come to some agree-

ment, as in the theory of functions of a complex variable, as to what is proper by way of initial limitations, either upon the kind of divergent series to be considered or upon the kind of sum formulas we shall employ in connection with them. Have we any good indications as to what such limitations may well be? Yes, but none that are universally accepted as yet. As to what the line of approach to this matter may properly be I shall have more to say eventually, but for the moment I deem it more proper that I attempt to answer another question which no doubt by this time has arisen in your own minds. How can it be that divergent series have come into prominence in the face of such conditions as I have just described, whereby there does not exist even to this day a strictly coherent and universally acceptable general theory of the subject? This is a very proper question, whose answer is twofold.

First, the various sum formulas, regardless of their interrelations and other such logical aspects, have been found to yield interesting information when applied to certain important special series, such as Fourier's series, Dirichlet's series, etc. For example, even though the Fourier series representing a given function $f(x)$ may be divergent at a certain point, still the series may be summable there by one or more formulas and the sum thus obtained may and usually does continue to serve fully as useful a purpose from the standpoint of mathematical physics or other applications as does the sum when the same series is convergent; that is, the sum in the extended sense furnishes the answer to the proposed physical problem. Extensive investigations have been carried out in this connection to determine sufficient conditions under which a Fourier series will be summable, analogous to the well-known conditions for its convergence, and similarly the problem has been carefully worked over for some of the other related developments such as those for an arbitrary function in terms of Bessel functions, or Legendre functions. In this connection it may be of interest to observe the central fact that whereas the Fourier series for $f(x)$ converges in general to the value

$$\frac{f(x - 0) + f(x + 0)}{2}$$

only in case $f(x)$ is of limited total fluctuation in the neighborhood of the point x under consideration, the same series will

be summable by the simplest of the well known sum formulas of Cesàro to this same value provided only that the right and left limits, namely $f(x + 0)$ and $f(x - 0)$, exist. Here we incidentally meet with an illustration of the manner in which summability includes convergence as a special case, since the latter set of conditions is evidently much less restrictive than those just mentioned for convergence. And it may be added that, so far as the Bessel expansions are concerned, much the same situation prevails as with Fourier series, but the case of the Legendre expansions is essentially different and all the more interesting because of its novelty. Here no new results follow so long as one uses the simplest of the Cesàro formulas; that is, so long as one uses the formula of order 1 these developments are summable under no less restrictive conditions than would insure convergence itself. But by using the same formula of order 2, or 1 higher, various new and interesting results follow. The manner in which the summable properties of the Legendre developments thus lie intermediate between the range of the formulas of orders 1 and 2 has led, it may be added, to a generalization of the whole conception of the formula to include fractional or even incommensurable orders of summation. Out of such a generalization there arise interesting special studies analogous to what we find in the ordinary elementary study of series. Thus, just as we know that in the case of the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

convergence merges into divergence as p passes through the value 1 from above, so we are able to determine for divergent series just where the critical order, or orders, are at which new information begins to be realized. Furthermore, the extended study of sum formulas simply upon their own merits, that is, without stressing their logical interrelations, has led very naturally to the notion of uniform summability corresponding to that of the uniform convergence of a series. And again, alongside of the same studies, corresponding studies have naturally arisen for divergent integrals. Here as before the underlying idea is that of setting up a formula which shall give the value of all convergent improper integrals and at the same time preserve a meaning and thus assign a value to some integrals that are divergent. As the formulas pertaining to such

studies are not as familiar as those for series, it may be proper to note that if the type integral be taken as

$$I = \int_a^\infty f(\beta) d\beta,$$

then the formula analogous to the Cesàro formula of order 1 and giving the value of I even in some cases of divergence is

$$I = \lim_{x \rightarrow \infty} \frac{1}{x} \int_a^x d\alpha \int_a^\alpha f(\beta) d\beta,$$

this formula having been first obtained, I believe, by Professor C. N. Moore in his thesis. It is equivalent to the somewhat simpler form

$$I = \lim_{x \rightarrow \infty} \int_a^x f(\beta) \left(1 - \frac{\beta}{x}\right) d\beta.$$

I shall not attempt any further details concerning sum formulas and the facts derivable either directly or by suggestion from them. We can certainly say, however, that this class of studies, as carried out independently for the various formulas, has greatly increased the range of interest in series in general and has enabled us to see convergent series in particular from new and very instructive points of view.

The second reason alluded to above for the prevalent interest in divergent series despite the lack as yet of any universally accepted general theory about them brings us to a certain very important aspect of the whole which we have not as yet mentioned—an aspect, it may be added, which was entirely disposed of in 11 lines of the encyclopedia article mentioned at the beginning. We refer to what is known as “asymptotic series.” This is in reality the oldest aspect which our subject presents. It may be said to have originated in an isolated note by Cauchy in 1843 relating to the well-known series of Stirling

$$(1) \quad \log \Gamma(x) = \frac{1}{2} \log 2\pi + (x - \frac{1}{2}) \log x - x \\ + \frac{B_1}{1 \cdot 2} \frac{1}{x} - \frac{B_2}{3 \cdot 4} \frac{1}{x^3} + \frac{B_3}{5 \cdot 6} \frac{1}{x^5} - \dots$$

($B_m = m$ th Bernoulli number).

Cauchy pointed out that this series, though divergent for all

values of x , may be used in computing $\log \Gamma(x)$ when x is large (and positive). In fact, he showed that, having fixed the number n of terms taken, the absolute error committed by stopping the summation at the n th term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small ($n > 3$) as x increases indefinitely. Cauchy's work on divergent series was confined, however, to the single series (1) and, owing to the overemphasis placed upon convergent processes by the successors of Cauchy and Abel, as mentioned earlier, no further progress was made in this field until the subject at last reappeared after more than forty years in connection with the researches of Poincaré upon the irregular solutions of linear differential equations. Poincaré considered those divergent series (normal series) of the form

$$(2) \quad e^{f(x)} x^\rho (A_0 + A_1/x + A_2/x^2 + \cdots);$$

$$f(x) = \text{a polynomial in } x, \quad \rho = \text{a constant,}$$

which for some time had been known to satisfy *formally* homogeneous linear differential equations of certain types having the point $x = \infty$ as a so-called "irregular point," and he showed essentially that in general to every such formal solution there corresponds an *actual* solution which can be represented by (2) in much the same sense as (1) was described above as representing $\log \Gamma(x)$. In view of the important significance of such results both from the standpoint of the possible use of divergent series and from that of the theory of differential equations, Poincaré set apart and discussed in some detail a broad class of divergent series of the special form (2), applying to them the name of "asymptotic series." Poincaré's results, however, in so far as they concerned differential equations, were noticeably incomplete, being limited by certain unfortunate restrictions, and thus his original studies have given rise in later years to numerous researches in which noteworthy advances have been made, though open questions in this connection still remain. Corresponding investigations, likewise begun by Poincaré, pertaining to linear difference equations have also been undertaken in recent years and carried to an advanced stage. Meanwhile another important aspect of the theory of asymptotic series has come into view; namely, that of actually determining the asymptotic developments of any given function—a problem of decided interest

for the study and classification of functions in general. I can perhaps employ the next few moments to no greater advantage than in sketching somewhat more completely these various fundamental problems relative to asymptotic series, pointing out incidentally certain unsolved special problems of noteworthy interest.

Referring again to differential equations, the studies in question may be said to center about the linear homogeneous differential equation

$$(3) \quad y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = 0,$$

wherein the coefficients $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$ are, in the simplest case, rational functions of x and, in the more extended case, are supposed to be developable in series of the form

$$(4) \quad a_r(x) = x^{rk} \left[a_{r,0} + \frac{a_{r,1}}{x} + \frac{a_{r,2}}{x^2} + \cdots \right] \quad (r = 1, 2, \cdots, n),$$

k being zero or a positive integer. In such a differential equation the point $x = \infty$ is in general an irregular point, so that the usual normal solutions are divergent series of the form (2). With reference to these solutions, we may now cite the following fundamental theorem:

"If for the equation (3) the roots $m_1, m_2, m_3, \cdots, m_n$ of the so-called characteristic equation, i. e., of the algebraic equation

$$(5) \quad m^n + a_{1,0}m^{n-1} + a_{2,0}m^{n-2} + \cdots + a_{n,0} = 0,$$

are distinct, equation (3) possesses n linearly independent solutions $y_1, y_2, y_3, \cdots, y_n$ which, for large values of x , are developable asymptotically in the form (2), wherein $f(x)$ is of degree $k + 1$, and $a_0 = 1$; that is, we have

$$(6) \quad y_r \sim e^{f_r(x)} x^{\rho_r} \left[1 + \frac{A_{1,r}}{x} + \frac{A_{2,r}}{x^2} + \cdots \right] \quad (r = 1, 2, 3, \cdots, n),$$

where $f_r(x)$ is a polynomial of degree $k + 1$ in x , while ρ_r is a constant."

If in this theorem the restriction be removed that the roots of the characteristic equation be distinct; that is, if multiple roots are present, the theorem fails and we at once encounter a problem for which no general solution has yet been obtained. Moreover, the theorem as just stated carries with it the assump-

tion that x is real. When x is regarded as complex, much the same results follow, the forms (6) holding now over certain sectors of the plane emanating from the origin, but here again much remains to be investigated, the existing theorems covering only what may be described as the simplest cases.

Contrasted with the same theorem for differential equations, is the corresponding fundamental theorem for linear difference equations:

"Given a homogeneous linear difference equation of the n th order, which we may write in the form

$$(7) \quad y(x+h) + a_1(x)y(x+h-1) + a_2(x)y(x+h-2) \\ + \cdots + a_n(x)y(x) = 0,$$

and let it be assumed that the coefficients $a_1, a_2, \dots, a_n$ are either rational functions of x or are developable in series of the form (4). Then, if the roots $m_1, m_2, \dots, m_n$ of the characteristic equation (5) are distinct and no one of them equal to zero, equation (7) possesses n linearly independent solutions $y_1, y_2, \dots, y_n$ valid for large positive values of x and developable asymptotically in the forms

$$(8) \quad y_r \sim [\Gamma(x+1)]^k m_r^x x^{\rho_r} \left[1 + \frac{A_{1,r}}{x} + \frac{A_{2,r}}{x^2} + \cdots \right] \\ (r = 1, 2, 3, \dots, n).$$

In case the characteristic equation presents multiple roots, or a zero root, no corresponding results appear to have been obtained, at least in general, though this whole subject has been interestingly discussed from an altogether different point of view and in a considerably larger measure of completeness by the introduction throughout of the so-called faculty series instead of the usual power series forms. Here again much remains to be done for the case of a complex variable, though a beginning corresponding to that cited above for differential equations has been made. The importance of these studies, both as regards differential and difference equations, lies, of course, in the fact that it is equations of these particular types that play a most fundamental rôle in analysis, both from the function theoretic standpoint and from that of applications. We shall not enter, however, into further details in this direction more than to mention the fact that corresponding studies

for non-homogeneous differential and difference equations have been considered but, like the homogeneous cases, are in but a limited state of completion.

As regards the problem of determining the asymptotic developments of a given function, which I also mentioned a moment ago, the meaning of this class of studies may perhaps be best understood from one or two simple illustrations. Let us take, for example, the following power series in which x is regarded as taking complex as well as real values:

$$(9) \quad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\rho - n}; \quad \rho = \text{any non-integral constant.}$$

The radius of convergence of this series is easily seen to be 1, so that the series itself yields no information as to the nature of the function $f(x)$ defined by it in the more distant portions of the plane. In order to secure such information and thus be able to follow the course of the function for values of x of large modulus it becomes necessary to develop $f(x)$ in some manner about the point infinity, as for example in power series in $1/x$, but the simple knowledge of the formula for the n th term of the given series (9) provides no immediate way of determining the coefficients of such a development. When once obtained, moreover, it may either converge or it may represent $f(x)$ only asymptotically. How actually to determine the development, whatever be its ultimate character, is the problem, and in the case before us it may be stated that it becomes

$$f(x) \sim -\frac{\pi(-x)^\rho}{\sin \pi \rho} - \frac{1}{(\rho+1)x} - \frac{2}{(\rho+2)x} - \dots,$$

this form holding at least so long as we confine ourselves to any sector of the plane which does not contain the positive half of the real axis. More generally, it may be shown that if we have any power series

$$\sum_{n=0}^{\infty} g(n)x^n,$$

wherein the coefficient $g(n)$ may be regarded as a function of a complex variable n and as such is analytic throughout the entire n plane except for a finite number p of poles, and at the same time, when considered for values of n of sufficiently large modulus, remains less than a constant, then the function $f(x)$

defined by this series will be developable in general throughout the distant portions of the plane either asymptotically or in a convergent series of the specific form

$$f(x) \sim - \sum_{m=1}^p r_m - \frac{g(-1)}{x} - \frac{g(-2)}{x^2} - \frac{g(-3)}{x^3} - \dots,$$

where r_m represents the residue of the function $\frac{\pi g(n)(-z)^n}{\sin \pi n}$ at the n th pole of $g(n)$. Corresponding results for various other type forms of power series have likewise been obtained, and again, similar studies have been extensively carried out for functions defined not by power series, but by altogether different though very important forms, such as infinite products, or faculty series. Sufficient has been said, I judge, on this aspect of asymptotic series so that you will perceive its bearing not only upon the determination of the values of a function in distant regions but also upon the broader problem of the classification of functions in general, since functions may clearly be distinguished from one another in classes corresponding to the different characters of their asymptotic developments. Very much remains to be done in this entire field of investigation.

What we have thus far said may be briefly summarized in the statement that the modern theory of divergent series contains essentially two branches, the first concerning the question as to how a sum may be assigned to a divergent series in general, and the second pertaining merely to the functional properties of that important special class of divergent power series known as asymptotic series. Of these two branches, the second, though characterized by theorems and results which usually bear a high degree of complexity, presents no logical inconsistencies and is thus in quite as satisfactory a state as convergent series themselves, while the first, or problem of summation, when considered as a whole is still in an unsatisfactory logical state because, as pointed out earlier, we have a large variety of sum formulas which, though agreeing with one another when applied to convergent series, fail to do so to a greater or less degree when applied to divergent series. However, all that remains in order to bring about perfect agreement everywhere is, as was also stated earlier, that we place proper limitations either upon the kind of diver-

gent series to be considered, or upon the kind of sum formulas to be retained, or both. In the time that remains to me I may therefore return for a few moments to the question as to how a logically consistent general theory of summation, if there is to be one, may well be constructed. It may be that the limitations which I am about to place may seem too restrictive to some, yet I take for granted that everyone shares with me the instinctive feeling that there should be a logically sound and fairly useful general theory of *some* sort and it is mainly in that spirit that my suggestions will be made.

In approaching the question let us first cast a glance back over the historical genesis of all the various sum formulas, for this is at once suggestive. The earliest and simplest of them is the one growing out of certain studies of Frobenius in 1880 relative to the behavior of the power series

$$(10) \quad \sum_{n=0}^{\infty} a_n x^n$$

for values of x upon its circle of convergence. His theorem in substance was as follows: "Suppose that the radius of convergence of (10) is 1, and let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$. Then, we shall have

$$(11) \quad \lim_{x \rightarrow 1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n \rightarrow \infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}$$

whenever the limit on the right exists." This is a straightforward result in the theory of functions having at first sight no relation to divergent series, nor indeed did it come to play any recognized part in the development of the latter for a considerable time. When it did it was because the left member of (11) is known to be identical with

$$(12) \quad \sum_{n=0}^{\infty} a_n$$

whenever this series is *convergent*, so it seemed natural in case it was divergent to continue assigning sums s to it in accordance with the formula

$$(13) \quad s = \lim_{n \rightarrow \infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}$$

so long as the limit on the right exists. This particular for-

mula for s found additional justification in the fact that the value it assigns to (12) when divergent is useful in the sense that it is the value of the function $f(x)$ defined by (10) at the point $x = 1$; that is, formula (13) furnishes the analytic continuation of the power series (10) at the point $x = 1$ upon its circle of convergence. It was essentially in this same spirit of useful as well as possible extension of idea that the sum formulas of Cesàro and Hölder, both of which contain (13) as a special case, were set up, as likewise the later transcendental sum formula of Borel wherein a definite integral is involved. These considerations immediately suggest that a logically coherent and at the same time useful theory of summability may be established by limiting ourselves throughout to those series (12) for which the corresponding power series (10) has a non-vanishing radius of convergence and furthermore limiting our use of sum formulas to those which, like the familiar ones of Cesàro and Borel, assign to a divergent series (12) a sum s which is equal to the value of the analytic continuation of this same power series (10) at the point $x = 1$. Such a value for s allows of no duplicity and is therefore unique, thus removing the primary logical defect heretofore mentioned in the present day aspect of the theory. Moreover, a theory thus limited in scope at once satisfies our desideratum of being useful, for it attaches itself in a most fruitful way to the very important subject of analytic continuation in the theory of functions of a complex variable. For example, it may be shown that in general any divergent power series is summable by Cesàro's formula at points upon its circle of convergence, thus furnishing the analytic continuation of the corresponding function at such points, but that it is not summable by this formula at points outside the same circle. On the other hand, if Borel's integral formula be used on the same series, it gives a sum and hence the analytic continuation not only for points upon the circle of convergence, but in certain regions lying outside this circle; namely, within the so-called polygon of summability formed by tangents to the circle at those points which are singular points of the function defined by the corresponding power series. Moreover, in a theory as thus restricted the usual rules for the manipulation and combination of convergent series are in large measure preserved. In short, we have left, it seems to me, a sufficient body of doctrine to be worthy of the name "general theory of summation."

I trust that I have now made clear my own feelings regarding the three questions raised at the outset; first, as to why divergent series have come into such prominence since the appearance of the early volumes of the encyclopedia, second, what has been done that really constitutes a vital advance and third, as to whether such series are at last upon a truly scientific basis. My only fear is that in attempting to couch the whole in very simple form I may have gone too far in this direction and thus violated a principle which, I believe it is said, the poet Browning always carefully observed; namely, of never using so simple a style that the intelligence of one's readers or hearers may be offended. But this is a rather treacherous principle, as most people discover in attempting to read Browning, so I may perhaps be pardoned if I have seemed to depart too far from it.

SOLUTIONS OF DIFFERENTIAL EQUATIONS AS FUNCTIONS OF THE CONSTANTS OF INTEGRATION.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society December 29, 1917.)

THE purpose of this note is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called "imbedding theorem" for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions.

Let C be an arc

$$(C) \quad x = u(\tau), \quad \tau_1 \leq \tau \leq \tau_2,$$

for which the function u is single-valued and has a continuous first derivative. The neighborhood C_ϵ of this arc is the totality of points (τ, x) which satisfy the inequalities

$$(C_\epsilon) \quad \tau_1 \leq \tau \leq \tau_2, \quad |x - u(\tau)| \leq \epsilon.$$

In the differential equation

$$(1) \quad \frac{dx}{d\tau} = f(\tau, x)$$

the function f is supposed

(a) to be single-valued and continuous in C_ϵ ;^{*}

(b) to satisfy the Lipschitz condition

$$(2) \quad |f(\tau, x) - f(\tau, x')| \leq \kappa |x - x'|$$

whenever (τ, x) and (τ, x') are both in C_ϵ ;

(c) to be such that the equation (1) has the arc C as a solution.

By a solution of equation (1) is always meant an arc of the type of C having a continuous derivative and satisfying the equation. The theorem to be proved is then the following:

For every neighborhood C_ϵ of the arc C with the properties just described there exists a second neighborhood C_δ through every point (τ_0, x_0) of which passes one and but one solution of equation (1), defined and in C_ϵ on the whole interval $\tau_1 \tau_2$. The function

$$x = v(\tau, \tau_0, x_0)$$

representing these solutions is continuous and has a continuous derivative $\partial v / \partial \tau$ in the region R of points (τ, τ_0, x_0) satisfying the conditions

$$(R) \quad \tau_1 \leq \tau \leq \tau_2, \quad (\tau_0, x_0) \text{ interior to } C_\delta.$$

If $f(\tau, x)$ has continuous partial derivatives of the n -th order in C_ϵ , then v and $\partial v / \partial \tau$ also have continuous partial derivatives up to and including those of order n when (τ, τ_0, x_0) lies in the region R .

The existence and continuity of the function $v(\tau, \tau_0, x_0)$ have been established in various ways. For the sake of completeness a proof will be given here which is based upon the method of approximation of Picard, and which makes use of the sequence of functions $\{v_m\}$ defined by the following equations:

^{*} If only the continuity of f with respect to τ is presupposed, then with the help of (b) it is provable that f is continuous in τ and x together. The proof for boundary points of C_ϵ is less direct than for interior points.

$$|v_2 - v_1| = \left| \int_{\tau_0}^{\tau} [f(\tau, v_1) - f(\tau, v_0)] d\tau \right|,$$

$$\leq 2\epsilon \frac{\kappa |\tau - \tau_0|}{1!}$$

since v_0 and v_1 both define curves in C_ϵ and hence differ by at most 2ϵ . An induction as before now gives

$$|v_{m+1} - v_m| \leq 2\epsilon \frac{\kappa^m |\tau - \tau_0|^m}{m!} \leq 2\epsilon \frac{\kappa^m \lambda^m}{m!},$$

showing that the series

$$(5) \quad v_1 + (v_2 - v_1) + \cdots + (v_{m+1} - v_m) + \cdots$$

converges uniformly.

From the last of the equations (3) it follows as usual that the limit function $v(\tau, \tau_0, x_0)$ satisfies the equation

$$v(\tau, \tau_0, x_0) = x_0 + \int_{\tau_0}^{\tau} f[\tau, v(\tau, \tau_0, x_0)] d\tau$$

and hence also the differential equation (1). If there were a second solution $w(t)$ of equation (1) through the point (τ_0, x_0) it would satisfy an equation similar to the last one, and consequently also the inequality

$$|w - v| \leq \left| \int_{\tau_0}^{\tau} \kappa |w - v| d\tau \right| \leq 2\epsilon \frac{\kappa |\tau - \tau_0|}{1!}$$

since 2ϵ is greater than $|w - v|$ on the interval $\tau_1\tau_2$. But successive application of this relation gives

$$|w - v| \leq 2\epsilon \frac{\kappa^m |\tau - \tau_0|^m}{m!},$$

which can be true only if v and w are identical, since the second member has the limit zero as m increases.

The proofs of the preceding paragraphs establish the existence, continuity, and uniqueness of the function $v(\tau, \tau_0, x_0)$, and also of its derivative $\partial v / \partial \tau$ since v satisfies (1). To prove that it has the further derivatives described in the theorem, suppose first that $f(\tau, x)$ has continuous first derivatives in C_ϵ . Then each of the functions v_m has a derivative

$$(6) \quad \frac{\partial v_m}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} f_x(\tau, v_{m-1}) \frac{\partial v_{m-1}}{\partial x_0} d\tau$$

and it is proposed to show that the sequence $\{\partial v_m/\partial x_0\}$ converges uniformly for values (τ, τ_0, x_0) in the region R . A well-known theorem* concerning the differentiation of a series term by term, applied to the series (5), then establishes the fact that the limit $v(\tau, \tau_0, x_0)$ of the sequence $\{v_m\}$ has as its derivative the limit of the sequence $\{\partial v_m/\partial x_0\}$.

Consider the two sums

$$(7) \quad B = 1 + \int_{\tau_0}^{\tau} f_x d\tau + \cdots + \int_{\tau_0}^{\tau} f_x \cdots \int_{\tau_0}^{\tau} f_x d\tau + \cdots,$$

$$(8) \quad \frac{\partial v_m}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} A_{m-1} d\tau + \cdots + \int_{\tau_0}^{\tau} A_{m-1} \cdots \int_{\tau_0}^{\tau} A_0 d\tau^m,$$

where $f_x = f_x(\tau, v)$ represents the first partial derivative of f with respect to x , and $A_n = f_x(\tau, v_n)$. The first sum is an infinite series, while the second has a finite number of terms found by successive application of formula (6). A simple inductive proof shows that each term of (8) approaches uniformly the corresponding term of (7) as a limit when m increases indefinitely, since the elements of the sequence $\{v_m\}$ approach uniformly the limit v . Further each term of both sums has absolute value less than the corresponding term of the series

$$1 + \frac{\kappa |\tau - \tau_0|}{1!} + \cdots + \frac{\kappa^n |\tau - \tau_0|^n}{n!} + \cdots$$

or of the series

$$(9) \quad 1 + \frac{\kappa \lambda}{1!} + \cdots + \frac{\kappa^n \lambda^n}{n!} + \cdots,$$

where κ is now the maximum of the absolute value of f_x in C . It follows that an integer n can be taken so large that all the terms of the series (7) after the n th have a sum with absolute value less than $\epsilon/3$, and also so large that the same is true for the expression (8). Since the individual terms of (8) approach uniformly those of (7) it is also true that for sufficiently large values of m the sums of the first n terms of (8) and (7) differ by less than $\epsilon/3$. Hence for such values of m the difference between $\partial v_m/\partial x_0$ and the sum of the series (7) is less than ϵ , and the derivatives $\partial v_m/\partial x_0$ approach the sum of the series (7) uniformly for all values (τ, τ_0, x_0) in the region R . The function v has therefore a derivative $\partial v/\partial x_0$ given by the series

* Goursat-Hedrick, A Course in Mathematical Analysis, vol. 1, p. 365.

$$(10) \quad \frac{\partial v}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} f_x(\tau, v) d\tau + \dots$$

$$+ \int_{\tau_0}^{\tau} f_x(\tau, v) \dots \int_{\tau_0}^{\tau} f_x(\tau, v) d\tau^m + \dots,$$

which converges uniformly, since its terms are respectively less in absolute value than those of (9), and $\partial v / \partial x_0$ is therefore continuous in the region R .

Similar reasoning shows the existence and continuity of the derivative $\partial v / \partial \tau_0$. For from equations (3)

$$\frac{\partial v_m}{\partial \tau_0} = -f(\tau_0, x_0) + \int_{\tau_0}^{\tau} A_{m-1} \frac{\partial v_{m-1}}{\partial \tau_0} d\tau$$

$$= -f_0 - \int_{\tau_0}^{\tau} A_{m-1} f_0 d\tau - \dots$$

$$- \int_{\tau_0}^{\tau} A_{m-1} \dots \int_{\tau_0}^{\tau} A_1 f_0 d\tau^{m-1} - \int_{\tau_0}^{\tau} A_{m-1} \dots \int_{\tau_0}^{\tau} A_0 u'(\tau_0) d\tau^m,$$

where $f_0 = f(\tau_0, x_0)$. The comparison series analogous to (9) is ν times that series, where ν is the larger of the maxima of $|f(\tau_0, x_0)|$ and $|u'(\tau_0)|$ in R , and the value of the derivative sought is the continuous sum of the uniformly convergent series

$$(11) \quad \frac{\partial v}{\partial \tau_0} = -f(\tau_0, x_0) - \int_{\tau_0}^{\tau} f_x(\tau, v) f(\tau_0, x_0) d\tau - \dots$$

$$- \int_{\tau_0}^{\tau} f_x(\tau, v) \dots \int_{\tau_0}^{\tau} f_x(\tau, v) f_0(\tau_0, x_0) d\tau^m - \dots.$$

Finally the equation

$$(12) \quad \frac{\partial v}{\partial \tau} = f(\tau, v)$$

shows that $\partial v / \partial \tau$ is continuous and has continuous first partial derivatives in the region R , since f and v have this property in C , and R , respectively. This completes the proof of the theorem for the case $n = 1$. The corresponding results for an arbitrary value of n are readily established by induction when the theorem for $n = 1$ has been proved for a system of equations instead of a single one, as will be shown below. In concluding the simpler case it is important to note the relation

$$(13) \quad \frac{\partial v}{\partial \tau_0} = - \frac{\partial v}{\partial x_0} f(\tau_0, x_0),$$

which follows readily from (11) and (10).

The proofs of the preceding theorems for a system of differential equations are of the same character as those above, but the equations used in the proofs will be much more complicated unless notations are used for matrices and multipartite numbers. Peano seems to have been the first to define and apply the moduli of multipartite numbers and matrices in proving existence theorems for differential equations, the case which he considered being that of a linear system.* If x denotes the set of numbers $(x', x'', \dots, x^{(p)})$, and f the set $(f', f'', \dots, f^{(p)})$, the single equation (1) will represent the system

$$(14) \quad \frac{dx^{(i)}}{dt} = f^{(i)}(t, x', \dots, x^{(p)}) \quad (i = 1, \dots, p).$$

By replacing absolute values everywhere by moduli, the modulus of x for example being

$$(15) \quad \text{mod } x = \sqrt{x'^2 + \dots + x^{(p)2}},$$

and using simple properties of multipartite numbers and matrices,† the proofs for the system (14) can be carried through quite simply and the notations will be the same, equation for equation, as in the paragraphs above. To facilitate the re-interpretation of the equations the following table of the symbols used is given:

Positive integers: i, j, m, n, p ;

Scalars: $\delta, \epsilon, \kappa, \lambda, \mu, \nu, \rho, \sigma, \tau, e$;

p -partite numbers: $a, f, x, y, z, u, v, w, \partial v / \partial \tau_0, \partial v_m / \partial \tau_0$;

p -square matrices: $A, B, f_x = \partial f / \partial x, \partial v / \partial x_0, \partial v_m / \partial x_0$.

In order to carry through the proofs of the existence and continuity of the derivatives of the p functions v , some additional properties of matrices are needed besides those given in the papers cited above. The definition of the modulus of a matrix given below is equivalent to that of Peano but different in form. It is closely associated with the notion of a limited

* *Mathematische Annalen*, vol. 32 (1888), p. 450.

† See also my paper, "The solutions of differential equations of the first order as functions of their initial values," *Annals of Mathematics*, 2d ser., vol. 6 (1905), p. 58.

matrix used by Hilbert in his theory of quadratic forms in a denumerable infinity of variables, and for the special case here considered is identical with the modulus as defined by E. H. Moore for the purposes of his General Analysis. It is applied here only to square matrices, but would be equally effective for those which are of unequal dimensions, and in particular for a matrix of one row and p columns it gives precisely the value (15). The table of properties in the next paragraph but one contains besides those of Peano some further properties which are useful in the present paper.

Let A be a p -square matrix, and let $\text{mod } A$ be the maximum of the bilinear form $Ay \cdot z^*$ on the set of values of y and z which satisfy the equations $\text{mod } y = \text{mod } z = 1$. This maximum is attained since the set over which y and z range is closed, and from the elementary theory of maxima and minima the values of y and z which determine the maximum satisfy the linear equations

$$(16) \quad Ay = \rho z, \quad \bar{A}z = \sigma y,$$

from which also

$$(17) \quad \bar{A}Ay = \rho\sigma y, \quad \bar{A}Az = \rho\sigma z,$$

where ρ, σ are scalars and $\bar{A}$ is the matrix formed from A by interchanging rows and columns. From the equations (16) one deduces readily

$$(18) \quad Ay \cdot z = \rho, \quad \bar{A}z \cdot y = Ay \cdot z = \sigma,$$

when $\text{mod } y = \text{mod } z = 1$, so that ρ and σ are both equal to the value of the modulus of A . Furthermore from the equations (16) and (18)

$$Ay \cdot Ay = \bar{A}Ay \cdot y = \rho^2,$$

which suggests that the modulus defined above may be identical with that of Peano, the value which he used being the square root of the maximum of $Ay \cdot Ay$ on the set of values y satisfying $\text{mod } y = 1$. By reasoning quite similar to that just used, and with the help of equations (17), this proves to be the case.

The modulus of a matrix has the following properties, which are readily provable:

* The expression $y \cdot z$ is the sum of the products of the elements of y and z ; Ay is the m -partite number whose elements are the dot-products of the rows of A by y . For the use of the dot in this connection see Gibbs-Wilson, Vector Analysis, p. 55.

- (1) $\text{mod } A = \text{maximum of } Ay \cdot z \text{ on the set of values } y, z \text{ satisfying } \text{mod } y = \text{mod } z = 1,$
 $= \text{maximum of } \sqrt{Ay \cdot Ay} \text{ on the set of values } y \text{ satisfying } \text{mod } y = 1;$
- (2) every element of A has absolute value $\leq \text{mod } A$;
- (3) $\text{mod } A < p^2\delta$ if each element of A has absolute value less than δ ;
- (4) $\text{mod } \kappa A = |\kappa| \text{mod } A$;
 $\text{mod } Ay \leq \text{mod } A \text{mod } y$;
 $\text{mod } AB \leq \text{mod } A \text{mod } B$;
- (5) $\text{mod } (A + B) \leq \text{mod } A + \text{mod } B$,
 $\text{mod } (A - B) \leq |\text{mod } A - \text{mod } B|$;
- (6) $\text{mod } A$ is a continuous function of the elements of A ;
- (7) $\text{mod } \int_{\tau_0}^{\tau_1} A d\tau \leq \left| \int_{\tau_0}^{\tau_1} \text{mod } A d\tau \right|$;
- (8) $|\lambda| < \text{mod } (dA/d\tau)$ for every derivative number λ of $\text{mod } A$.

In the statement (7) the elements of A are supposed to be integrable functions of τ on the interval $\tau_0 \leq \tau \leq \tau_1$, and in (8) they are supposed to have unique derivatives at the value of τ considered. The property (8) is not necessary for the applications of the matrix theory in the present paper.

The property (2) holds because, by choosing all elements of y and z to be zero except one element equal to $+1$ or -1 in each, the value $Ay \cdot z$ may be made equal to the absolute value of a selected one of the elements of A . In the expression $Ay \cdot z$ with $\text{mod } y = \text{mod } z = 1$ there are p^2 terms each consisting of an element of A multiplied by two numbers with absolute values ≤ 1 . Hence when each term of A has absolute value less than δ the maximum of $Ay \cdot z$ surely does not exceed $p^2\delta$. The proof of the first formula (4) is immediate. The second part follows since for every y

$$\sqrt{Ay \cdot Ay} = \sqrt{\frac{Ay \cdot Ay}{(\text{mod } y)^2}} \text{mod } y \leq \text{mod } A \text{mod } y.$$

To prove the third part of (4), suppose that y, z are sets of values having $\text{mod } y = \text{mod } z = 1$ and giving $ABy \cdot z$ its maximum. Then

$$\text{mod } AB = ABy \cdot z = \frac{ABy \cdot z}{\sqrt{By \cdot By}} \sqrt{By \cdot By} \leq \text{mod } A \text{mod } B.$$

If y and z are similarly the values giving $(A + B)y \cdot z$ its maximum, then

$$\begin{aligned}\text{mod } (A + B) &= (A + B)y \cdot z \\ &= Ay \cdot z + By \cdot z \leq \text{mod } A + \text{mod } B.\end{aligned}$$

The second part of (5) follows from the first since

$$\begin{aligned}\text{mod } A &\leq \text{mod } (A - B) + \text{mod } B, \\ \text{mod } B &\leq \text{mod } (B - A) + \text{mod } A.\end{aligned}$$

If $A + \Delta A$ is a matrix formed from A by giving its elements increments with absolute values less than δ , then from the second property (5), and (3),

$$|\text{mod } (A + \Delta A) - \text{mod } A| \leq \text{mod } \Delta A < p^2\delta,$$

which proves the statement (6). The property (7) is readily proved by applying the first formula (5) to the sum whose limit is the definite integral on the left, and then taking the limit. Finally the inequality

$$\left| \frac{\text{mod } (A + \Delta A) - \text{mod } A}{\Delta\tau} \right| \leq \text{mod } \frac{\Delta A}{\Delta\tau}$$

is deducible with the help of the second property (5), and it justifies (8) since every derivative number of $\text{mod } A$ is the limit of the fraction on the left when $\Delta\tau$ approaches zero over a suitably selected sequence of values.

In the equation (6) for the system (14) the symbol $\partial v_m / \partial x_0$ represents the matrix of derivatives $|| \partial v_m^{(i)} / \partial x_0^{(j)} ||$ ($i, j = 1, \dots, p$) and the first term on the right is to be interpreted as the identity matrix I . A similar agreement holds for the first terms of (7), (8), and (10). The symbol f_x now represents the matrix of derivatives $|| \partial f^{(i)} / \partial x^{(j)} ||$. Hence in (6) and the three last mentioned equations every term is a p -square matrix. With the help of the matrix properties (7) and (4₂) above

$$\begin{aligned}\text{mod } \int_{\tau_0}^{\tau} A_{m-1} \cdots \int_{\tau_0}^{\tau} A_{m-n} d\tau^n \\ \leq \left| \int_{\tau_0}^{\tau} \text{mod } A_{m-1} \cdots \int_{\tau_0}^{\tau} \text{mod } A_{m-n} d\tau^n \right| \leq \frac{\kappa^n |\tau - \tau_0|^n}{n!},\end{aligned}$$

where κ is the maximum of $\text{mod } f_x$ in the region C_* . Hence the terms of the series (9) exceed the moduli of the respective

terms in (7) and (8). An integer n can now be taken so large that the modulus of the sum of the terms of (8) after the n th is less than $\epsilon/3$, and so that the same is true for the sum of the terms after the n th in the series (7). Since the elements of the sequence $\{v_m\}$ converge uniformly to the elements of v , the elements of the matrix represented by the first n terms of the sum (8) converge to the corresponding elements of the similar matrix from the series (7), and by an application of the matrix property (3) it is clear that the modulus of the difference of these matrices is less than $\epsilon/3$ when $m > n$ is sufficiently large. For such values m therefore

$$\text{mod} \left(\frac{\partial v_m}{\partial x_0} - B \right) < \epsilon,$$

and because of the matrix property (2) each element of $\partial v_m / \partial x_0$ converges uniformly, as m increases, to the respective element of B . Hence every one of the elements of v has a derivative with respect to every element of x_0 and the matrix $\partial v / \partial x_0$ of these derivatives is B . Finally the series of moduli of the terms of (10) converges uniformly, implying also, on account of property (2), the uniform convergence of every series of corresponding elements in the matrices which the terms of (10) represent, and showing therefore that every element of the matrix $\partial v / \partial x_0$ is a continuous function.

The proof of the existence and continuity of the p derivatives $\partial v / \partial \tau_0$ is so similar that it is unnecessary to exhibit it in detail.

The proofs which have been given above establish the properties of the solution $v(\tau, \tau_0, x_0)$ of a system of equations for the case $n = 1$ described in the theorem, and the proof can now be given for the general case by an induction. Suppose that the existence and continuity of the n th derivatives of v and $\partial v / \partial \tau$ have been proved to be a consequence of the assumption that f has continuous n th derivatives, and suppose further that the functions f have continuous derivatives in C , of order $n + 1$. The system of $2p + 1$ equations

$$(19) \quad \frac{d\tau_0}{d\tau} = 0, \quad \frac{dx_0}{d\tau} = 0, \quad \frac{dw}{d\tau} = f_x(\tau, v)w$$

in the variables τ, τ_0, x_0, w has the solution

$$(\Gamma) \quad \tau_0 = \text{const.}, \quad x_0 = \text{const.}, \quad w = \frac{\partial v}{\partial x_0} w_0$$

with the initial values τ_0, x_0, w_0 for $\tau = \tau_0$, as one readily sees by differentiating equation (12) for the elements of x_0 and from the fact that $\partial v / \partial x_0$ is the identity matrix when $\tau = \tau_0$. There is a neighborhood Γ , in which the second members of equations (19) have continuous n th derivatives, provided that (τ, τ_0, x_0) is in R , since v has this property and f has by hypothesis continuous derivatives of order $n + 1$. Hence there is also a region P for the solution Γ , analogous to R for C , in which the solution Γ has continuous n th derivatives. This shows that all of the elements of the matrix $\partial v / \partial x_0$ have continuous n th derivatives since w_0 is an arbitrary constant multipartite number. The same is true of the elements of the multipartite function $\partial v / \partial \tau_0$ from equations (13), and for the elements of $\partial v / \partial \tau$ from equations (12). Hence all of the derivatives of v of order $n + 1$ exist and are continuous as described in the theorem. That $\partial v / \partial \tau$ has the same property follows at once from the relations (12).

In conclusion it may be remarked that the theorems concerning the dependence of the solutions v upon parameters involved in the equations (14) require no proofs essentially different from those given above. A system of the form

$$\frac{dx}{d\tau} = f(\tau, x, a),$$

where a is a multipartite set of parameters, may in fact be replaced by the system

$$\frac{dx}{d\tau} = f(\tau, x, a), \quad \frac{da}{d\tau} = 0$$

in the variables τ, x, a . If the functions f have continuous derivatives of order n in τ, x, a , then the theorem proved above justifies the statement that the derivatives of order n for the solutions $x = v(\tau, \tau_0, x, a)$ and their derivatives v_τ will also exist and be continuous in the region analogous to R for these equations.

INVOLUTIONS ON THE RATIONAL CUBIC.

BY PROFESSOR R. M. WINGER.

(Read before the San Francisco Section of the American Mathematical Society October 27, 1917.)

Introduction.

1. THE general subject of involution as applied to rational curves has been widely studied, notably by Weyr, Stahl, Coble and many Italian writers. It is the purpose of this paper to discuss certain involutions on the rational cubic, R^3 .

If s_i denote the elementary symmetric functions of coordinates $x_1, x_2, \dots, x_n$ of n points (elements in the binary domain), the most general involution of order n , $I_{n-1,1}$, i. e., one in which $n - 1$ points of a set determine the remaining one, will be defined by

$$(1) \quad a_0 s_n + a_1 s_{n-1} + a_2 s_{n-2} + \dots + a_{n-1} s_1 + a_n = 0.$$

The involution is thus made up of all sets of n points apolar to a fixed set, the n -fold points of the involution, given by

$$(2) \quad a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \binom{n}{2} a_2 x^{n-2} + \dots + \binom{n}{n-1} a_{n-1} x + a_n = 0.$$

The following alternative and equivalent definition is serviceable when the n points of a set are represented implicitly by an equation: An $I_{n-1,1}$ is an $(n - 1)$ -parameter family of binary forms of order n

$$(3) \quad f_0 + k_1 f_1 + k_2 f_2 + \dots + k_{n-1} f_{n-1}.$$

More generally, if $n - r$ points of a set suffice to determine the remaining r , x_i must satisfy r equations of the type (1) and (3) reduces to an $(n - r)$ -parameter family. The corresponding involution is denoted by $I_{n-r,r}$.

2. Choosing for triangle of reference the nodal tangents and the line of flexes, the equation of the curve may be written in the canonical form

$$(4) \quad x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 + 1.$$

The peculiar adaptability of the language of involution to describe the properties of the curve will appear from a few examples. Thus the condition that three points be on a line is

$$(5) \quad s_3 + 1 = 0,$$

i. e., (5) defines the involution which comprises all line sections. If the line is tangent at τ , meeting again at t , the equation becomes

$$(6) \quad \tau^2 t + 1 = 0.$$

Again (from (6)) *the parameters of contacts of tangents from an arbitrary point of R^3 belong to the quadratic involution $s_1 = 0$ whose double points are the nodal parameters.**

Finally, *lines joining pairs of points in a quadratic involution envelop a conic perspective to R^3 .†*

The line $\overline{t_1 t_2}$ is

$$(7) \quad (s_1 s_2 - 1)x_1 + (s_1 - s_2^2)x_2 - 3s_2 x_3 = 0.$$

If the double points of the involution are given by

$$(8) \quad at^2 + 2bt + c = 0,$$

the equations of the perspective conics in lines, found by requiring that (8) be apolar to (7) considered as a quadratic in t_2 , are

$$(9) \quad u_1 = bt^2 - ct + a, \quad u_2 = ct^2 - at + b, \quad u_3 = -3bt.$$

These conics have each three contacts with R^3 which belong to an $I_{2,1}$, $s_3 = 1$, whose triple points are the sextactic points.‡

Among the tri-tangent conics must be counted the degenerate conics consisting of two tangents from a point of the curve. Thus the point t and the contacts of tangents from t are a set apolar to the sextactic points. The equation of any composite conic, e. g., the pair of tangents from t_1 , is found at once by taking the discriminant of (7) considered as a quadratic in t_2 .

Since the triple points are a set in the involution, there is one conic touching at the sextactic points. This is a remarkable conic which we shall call N . It is obtained by requiring

*This theorem, discovered independently, is referred to by Weyr, *Wiener Berichte*, vol. 79 (1879), p. 429 ff., as known.

†Coble, "Symmetric binary forms and involutions," *Amer. Jour. Math.*, vol. 32, p. 358.

‡The equations of the three sextactic conics are given by (9) when $b = 1$; $a = c = -3, -3\omega, -3\omega^2$, ($\omega^3 = 1$).

that the double points (8) of the quadratic involution be the nodal parameters. In this case $a = c = 0$ and the conic is

$$(10) \quad \begin{aligned} u_1 &= t^2, \quad u_2 = 1, \quad u_3 = -3t, \quad \text{or} \quad u_3^2 - 9u_1u_2 = 0, \\ x_1 &= 3, \quad x_2 = 3t^2, \quad x_3 = 2t, \quad \text{or} \quad 9x_3^2 - 4x_1x_2 = 0. \end{aligned}$$

Combining the foregoing discussion with a theorem stated at the beginning of this section, we may say: *The lines joining pairs of contacts of tangents from the points of R^3 envelop a conic N which touches the nodal tangents (where they meet the line of flexes) and has contacts with R^3 at the sextactic points.*

3. As an example of an $I_{1,2}$ may be mentioned the involution set up by the pencil of lines $(ux) + \lambda(vx) = 0$. The lines will cut out a pencil, i. e., an $I_{1,2}$ of binary cubics, say $u + \lambda v$. The contacts of tangents from the center of the pencil which are the double points of the involution are given by the Jacobian J of u and v .*

Are there any lines u whose cubi-covariant points are also line sections u' ? If so, these lines may be taken as the base lines of a pencil and the involution becomes $u + \lambda u'$. That is, from the intersection P of u and u' can be drawn not four tangents but a pair of repeated tangents. This can happen if and only if P is the intersection of two flex tangents. Hence if u is a line of a pencil with center P , its cubi-covariant points lie on a line u' of the same pencil. *In other words the locus of lines u whose cubi-covariant points lie on a line u' consists of the vertices of the triangle of flex tangents, while u' envelops the same points.*

Lines u and u' in any pencil P belong to a quadratic involution of lines whose double lines are the two flex tangents meeting at P .

The points P are $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ and are therefore fully perspective with the reference triangle.

Other Contact Conics.

4. To find the intersections of R^3 with a general conic we substitute equations (4) in the trilinear equation of the conic. The result will be a sextic in t , in which obviously the coefficient of the highest power is the same as the constant term. Hence the necessary and sufficient condition that six points lie on a conic is that their parameters satisfy the equation

* Salmon, Higher Algebra, fourth edition, p. 162.

$$(11) \quad s_6 = 1,$$

or that they belong to a sextic involution $I_{5,1}$.

We shall get contacts when two or more t 's come together.* Passing to the extreme case, suppose all six t 's coincide. Then

$$(12) \quad t^6 - 1 = (t^3 + 1)(t^3 - 1) = 0,$$

which includes the flexes among the sextactic points. The conics then degenerate of course into the flex tangents repeated.

5. Next suppose five points coincide at τ . Then

$$(13) \quad \tau^5 t - 1 = 0,$$

which says that at each point τ of R^3 there is a conic with 5-point contact,† but through a given point t five such conics pass. Moreover the product s_5 of the five quintactic parameters satisfies the equation $s_5 = 1/t$ (from (13)). Hence, by (11), *the quintactic points of the five quintactic conics which pass through t (simply) lie on a conic with t .*

6. Let four intersections coincide at τ and two at t , or

$$(14) \quad \tau^4 t^2 = 1.$$

There would seem to be four conics with simple contact at t and with 4-point contact elsewhere. But among these are the tangent lines from t each counted twice and therefore only two proper conics with contacts satisfying the equation

$$(15) \quad \tau^2 t = 1.$$

Or the quartactic points τ of conics touching at t are given by

$$(16) \quad \tau^2 - 1/t = 0.$$

Hence they lie on a line with t . Moreover they are harmonic with the contacts of tangents from t . They are likewise harmonic with the nodal parameters; hence the line joining them is a line of conic N and their tangents meet again on the curve, viz., at $-t$. Again $-t$ is on a line with the contacts of tangents from t . These statements apply equally well to $-t$.

* Contact here simply means coincident intersections and will include improper contact as well as ordinary tangency, i. e., coincidence of consecutive parameters.

† We shall call this a *quintactic* point and the conic a *quintactic* conic, adopting similar terms for the other cases.

We thus have a striking configuration which may be described as follows:

From a point $(-1/t^2)$ of R^3 draw the two tangents t and $-t$ and from each of these points the pair of tangents. The two pairs of contacts of tangents from t and $-t$ are harmonic and form therefore with the nodal parameters three mutually harmonic pairs. The line joining contacts of tangents from t passes through $-t$ and is therefore a tangent from $-t$ to conic N , and vice versa. The other tangent to N from either point is the junction of the two. At either point (t or $-t$) two conics touch which have quartactic points at contacts of tangents from the other.

7. If now the six intersections coincide in two triples, the equation becomes

$$(17) \quad t^3 \tau^3 = 1, \quad \text{or} \quad \tau^3 - 1/t^3 = 0.$$

Hence at each point t of R^3 there are three osculating conics which osculate the curve again at points $1/t$, ω/t , ω^2/t respectively. At each of these points there are three osculating conics which osculate again, one each at t , ωt and $\omega^2 t$. We have thus a configuration of six points and nine conics. If the points are arranged in two rows, as

$$\begin{array}{ccc} t & \omega t & \omega^2 t \\ 1/t & \omega/t & \omega^2/t \end{array}$$

then at each point in either row three conics osculate, each of which osculates again at one point of the other row. The six points themselves lie on a tenth conic whose equation is

$$9t^3 x_3^2 - (t^3 + 1)^2 x_1 x_2 = 0.$$

Moreover the two triangles as written are harmonically perspective from each of the flexes.

8. Finally we obtain tri-tangent conics when the points of (11) coincide in pairs. The contacts then satisfy the equation

$$s_3^2 = t_1^2 t_2^2 t_3^2 = 1.$$

But if the right side is -1 , the conic is any line section repeated. The contacts of all tri-tangent conics (except repeated lines) therefore belong to the $I_{2,1}$, $s_3 = 1$. Since this is the same involution as that defined by the sextactic points we infer that the perspective conics (9) Art. 2 embrace all tri-tangent conics.

Hyperosculating Curves.

9. The foregoing method can be applied to the study of other contact curves of the R^3 . We shall restrict our attention, however, to those which have complete intersection at a point. Such curves of order n will be called *hyperosculating curves*, designated by H_n . The contacts, denoted by P_{3n} , will be called *hyperosculating points*. The simplest of these curves are the flex tangents, of course, which taken n times must be reckoned among all H_n 's. Similarly the sextactic conics counted n times will be included among the H_{2n} 's of even degree, etc.

If now the x 's from (4) are substituted in a general ternary equation of degree n , there results a binary equation of degree $3n$ in t , wherein the coefficient of the highest power of t differs at most in sign from the constant term. Denoting by s_{3n} the product of the roots of this equation, we have as the condition that $3n$ points be the complete intersections of R^3 and a curve of degree n

$$(18) \quad s_{3n} = (-1)^n.$$

i. e., that they belong to an involution, $I_{3n-1,1}$.

Hence the contacts, P_{3n} , of hyperosculating curves are given by

$$(19) \quad t^{3n} = (-1)^n.$$

10. There will be two cases according as n is odd or even.

Case I. n odd. When $n = 1$ we have the points of inflexion. That is, there are 3 H_1 's whose P_3 's lie on a line.

When $n = 3$, (19) is

$$(20) \quad t^9 + 1 = (t^3 + 1)(t^6 - t^3 + 1).$$

There are thus 9 H_3 's whose P_9 's are on a cubic. But three of these P_9 's are the flexes, counted three times, and are therefore the complete intersections of a line and R^3 . Hence there are 6 proper H_3 's whose contacts are on a conic.

In general, n odd, the contacts of H_n 's are given by

$$(21) \quad t^{3n} + 1 = 0.$$

Here $s_{3n} = -1$. Hence there are $3n$ points at which H_n 's, including degenerate cases, can be drawn. These points are the complete intersections of R^3 and a C_n . This C_n , however,

is always composite, since it contains the line C_1 of flexes. It may contain also other factors C_{k_i} at whose intersections can be drawn degenerate H_n 's which are $H_{n/r}$'s of lower order repeated r times.

If $C_n = C_1 C_{k_1} C_{k_2} \dots C_{k_j} C_{n'}$, it follows that *there are $3n'$ P_{3n} 's at which proper H_n 's can be drawn and these points are the complete intersections of R^3 and $C_{n'}$* . For the lower values of n the C_n 's are appended, from which can be inferred the number of P_{3n} 's at which proper H_n 's can be drawn; e. g., there are 12 P_{15} 's at which proper H_5 's can be drawn and these points are on a quartic C_4 .

$$(22) \quad \begin{aligned} C_3 &= C_1 C_2, & C_5 &= C_1 C_4, & C_7 &= C_1 C_6, & C_9 &= C_1 C_2 C_6, \\ C_{11} &= C_1 C_{10}, & C_{13} &= C_1 C_{12}, & C_{15} &= C_1 C_2 C_4 C_8. \end{aligned}$$

11. *Case II.* n even and equal to $2m$. The hyperosculating points are now given by

$$(23) \quad t^{6m} - 1 = (t^{3m} - 1)(t^{3m} + 1).$$

There are two cases according as m is odd or even.

(a) When m is odd the second factor of (23) gives points P_{6m} which are P_{3m} taken twice; while the other factor

$$(24) \quad t^{3m} - 1 = (t^3 - 1)(t^{3m-3} + t^{3m-6} + \dots + t^3 + 1).$$

The first factor of (24) corresponds to the sextactic points which are to be taken m times. *The other factor indicates that there are $(3m - 3)$ P_{3n} 's which lie on a C_{m-1} .*

(b) When m is even the first factor of (23) names contacts of (H_m) 's². *The other factor says that there are $3m$ P_{3n} 's which lie on a C_m .* These curves C_{m-1} and C_m may or may not be composite, but as above we can infer that the points at which proper H_n 's can be drawn are the complete intersections of R^3 and a C_m .

In particular if n is a power of 2, say $n = 2^a$, (19) becomes

$$(25) \quad (t^{3 \cdot 2^a} - 1) = (t^{3 \cdot 2^{a-1}} - 1)(t^{3 \cdot 2^{a-1}} + 1) = 0,$$

the first factor of which names contacts of curves $H_{2^{a-1}}$ repeated, except when $a = 1$. The second factor gives proper $P_{3 \cdot 2^a}$'s which by (18) lie on a $C_{2^{a-1}}$. Moreover in virtue of relation (6) these $P_{3 \cdot 2^a}$'s are contacts of tangents from proper $P_{3 \cdot 2^{a-1}}$'s. We have thus the following chain theorem: *The*

3 sextactic points are contacts of tangents from the flexes P_3 . The 6 contacts of tangents from the sextactic points are the points P_{12} . The 12 contacts of tangents from P_{12} in turn are the points P_{24} , and so on ad infinitum.

UNIVERSITY OF OREGON.

RELATED INVARIANTS OF TWO RATIONAL SEXTICS.

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society September 4, 1918.)

LET the parametric equations of the R_3^6 , the rational curve of order six in three dimensions, be

$$(1) \quad x_i = \delta_{it}^6 \equiv a_i t^6 + 6b_i t^5 + 15c_i t^4 + 20d_i t^3 + 15e_i t^2 + 6f_i t + g_i \quad (i = 1, 2, 3, 4),$$

and let the parametric equations of the R_2^6 , the rational plane curve of order six, be of the form

$$\begin{aligned} x_1 &= \alpha_t^6 \equiv a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6, \\ x_2 &= \beta_t^6 \equiv a' + b't + c't^2 + d't^3 + e't^4 + f't^5 + g't^6, \\ x_3 &= \gamma_t^6 \equiv a'' + b''t + c''t^2 + d''t^3 + e''t^4 + f''t^5 + g''t^6. \end{aligned}$$

It is well known that all plane sections of the R_3^6 are apolar to a doubly infinite system of binary sextics, and that all line sections of the R_2^6 are apolar to a triply infinite system of binary sextics. We shall let the four binary sextics δ_{it}^6 of (1) be four linearly independent sextics of the apolar system of the R_2^6 , and the $\alpha_t^6, \beta_t^6, \gamma_t^6$ of (2) be three linearly independent sextics of the apolar system of the R_3^6 . Our purpose is to point out briefly the relation between the invariants of the R_2^6 and the invariants* of the R_3^6 .

By means of the twelve equations

* This relation must not be confused with the correspondence between invariants of the R_2^n and covariant surfaces of the R_3^n .

$$\begin{aligned}
 & a_i a - b_i b + c_i c - d_i d + e_i e - f_i f + g_i g = 0, \\
 (3) \quad & a_i a' - b_i b' + c_i c' - d_i d' + e_i e' - f_i f' + g_i g' = 0, \\
 & a_i a'' - b_i b'' + c_i c'' - d_i d'' + e_i e'' - f_i f'' + g_i g'' = 0 \\
 & (i = 1, 2, 3, 4),
 \end{aligned}$$

it may be easily proved that the four-rowed determinants of the matrix of the coefficients of δ_i^6 of the type $|abcd|$ are proportional to the complementary three-rowed determinants of the matrix of the coefficients of $\alpha_i^6, \beta_i^6, \gamma_i^6$ of the type $|ef'g''|$. Let T denote the substitution of the three-rowed determinants of (2) for the proportional four-rowed determinants of (1), and T^{-1} the inverse substitution.

Invariants of the R_3^6 are combinants of the four sextics δ_i^6 , and conversely, and these are rationally expressible in terms of the determinants of the type $|abcd|$. Invariants of the R_2^6 are combinants of $\alpha_i^6, \beta_i^6, \gamma_i^6$, and conversely, and these are rationally expressible in terms of the determinants of the type $|ab'c''|$. The combinants of δ_i^6 are implicit invariants of the R_2^6 which become explicit invariants of the R_2^6 after the application of T . Similarly, combinants of $\alpha_i^6, \beta_i^6, \gamma_i^6$ are implicit invariants of the R_3^6 which are transformed into explicit invariants of the R_3^6 by means of T^{-1} . Hence any explicit invariant I of the R_3^6 is transformed into an explicit invariant I' of the R_2^6 by means of T . Similarly, $T^{-1} I' = I$. It is evident that the order of I in the $|abcd|$ is the same as that of I' in the $|ab'c''|$. We shall now mention a few illustrations of this relation.

If U' is the undulation invariant of the R_2^6 , $T^{-1} U' = U$ is the stationary line invariant of the R_3^6 . From P , the pentatactic plane invariant of the R_3^6 , we obtain $TP = P'$, the cusp invariant of the R_2^6 . Similarly, from Q , the quinquese-cant line invariant of the R_3^6 , we derive $TQ = Q'$ whose vanishing defines an R_2^6 such that any six of its collinear points have parameters apolar to a binary quintic. If $N = 0$ is the necessary and sufficient condition that the R_3^6 have a node, $TN = N' = 0$ defines an R_2^6 which has one secant that cuts out a cyclotomic set of parameters.

PENNSYLVANIA STATE COLLEGE,
May, 1918.

IN MEMORIAM: ELLERY WILLIAM DAVIS.

BY PROFESSOR E. R. HEDRICK

(Read at the Chicago meeting of the American Mathematical Society
April 12, 1918.)

AT the invitation of the committee on the programme I have the honor to pay brief tribute to the friend and counsellor of many of you, and of mine, our late colleague, Ellery William Davis. It is fitting indeed that this Society pause in the midst of our scientific session to mark the passing of such a life.

He was a mathematician of insight and power. His published works contain many titles of lasting interest. Among those of technical mathematical interest may be noted particularly the following:

1. "An expression of the coordinates of a point on a binodal quartic curve as rational functions of the elliptic functions of a variable parameter." *American Journal of Mathematics*, volume 5 (1882), pages 333-336.
2. "Geometrical illustrations of some theorems in number." *American Journal of Mathematics*, volume 15 (1893), pages 84-90.
3. "A note on the invariance of the factors of composition of a group." *American Journal of Mathematics*, volume 19 (1897), page 191.
4. "On the sign of a determinant's term." *American Journal of Mathematics*, volume 19 (1897), page 383.
5. "Note on special regular reticulations." *Bulletin of the American Mathematical Society*, volume 4 (1897-1898), pages 529-530.
6. "The group of the trigonometric functions." *Bulletin of the American Mathematical Society*, volume 5 (1898-1899), pages 380-381.
7. "Some groups in logic." *Bulletin of the American Mathematical Society*, volume 9 (1902-1903), pages 346-348.

Other papers, less well known, which have appeared in journals not wholly mathematical are:

8. "A definition of mathematical probability." Baldwin's *Encyclopedia of Philosophy and Psychology*, 1902, pages 344-353.

9. "The elliptic functions and the general symmetric group in four letters." *Nebraska University Studies*, volume 4 (1904), pages 231-247.
10. "The imaginary in geometry." *Nebraska University Studies*, volume 10 (1910), pages 1-58.
11. "Cantor's leap into the transfinite." *Mid-West Quarterly*, volume 4 (1917), pages 239-250.

Among other publications, two of his books have exerted influence on mathematicians and the teaching of mathematics. One of these, "The Logic of Algebra," should find a place in every mathematical library. This he himself regarded as his most important work and it is certainly worthy of a lasting place in mathematical literature.

Finally his text on the Calculus is one which is peculiarly stimulating and will probably affect the teaching of that subject for a long time.

Though this work was good, it alone would not account for the high esteem in which he was held. One lesson to us all is that his gift for friendship, his warm humanism, his broad sympathies, even more than his direct mathematical productivity, are the bases of that strong feeling of respect and admiration which he so generally inspired. I have seen a number of notices of his passing which I should be glad to read to you, but I shall leave them with the remark that men of all classes have felt what I have expressed in this paragraph.

I might quote from two letters which I received after the programme for this meeting was printed. One of them from a well known woman says: "I am impelled to write to you to say that I count my three years experience in teaching in his department a very rich memory. . . . In particular, I should emphasize his pains in giving a young instructor free scope in whatever field the instructor might be interested. He was unselfish and a very generous spirit." The other is from a man recognized as among the first of American mathematicians. He says: ". . . I feel like saying that few other American mathematicians have encouraged me more by their sympathetic interest than Ellery Williams Davis did."

One illustration of his keenness of vision is now particularly noteworthy, his attitude toward this war and toward Prussianism. Early in the struggle in 1914 he voiced the opinion that this was our war and that the defeat of Prussia was essential to the preservation of liberty and democracy throughout

the world. To us this is significant as a striking instance of his vision and of his unhesitating support of right and high morality regardless of the immediate effects upon his own personal interests. In mathematical circles the same fearlessness of vision has led him to support warmly those projects or innovations which seemed dangerous to the more timorous. I may mention his attitude toward maintaining requirements in mathematics where real grounds exist and for abolishing the requirements otherwise. His attitude toward those reforms which started with the Perry movement and which have since broadened to much wider scope is well known. Finally, his support and sympathetic interest in the Mathematical Association of America is known to all who were interested in that movement.

Ellery Williams Davis was born in Oconomowoc, Wisconsin, on March 29, 1857. He died in Lincoln, Nebraska, Sunday, February 3, 1918. He was graduated with the degree of A.B. at the University of Wisconsin and with the degree of Ph.D. at Johns Hopkins in 1884. On June 20, 1886, he was married to Miss Annie T. Wright, who with four sons and a daughter survives him.

He was professor of mathematics at Florida Agricultural College from 1884 to 1886, at South Carolina College from 1886 to 1893, and at the University of Nebraska from 1893 to 1918. He was also dean of the department of liberal arts at Nebraska from 1901 to 1918.

In closing, let me put into words sentiments that I know are in the hearts of all who knew him. In his death we feel a personal sense of loss and we believe that university spirit and mathematical science in this country, which he so loyally upheld and furthered, have suffered through his passing.

A CORRECTION.

I WISH to call attention to the following errata in my note, "Some theorems of comparison and oscillation" in the April BULLETIN.

(1) On page 331, among the conditions under Theorem I, " $\alpha_2(x)/\alpha_1(x)$ never decreases" should read " $-\alpha_2(x)/\alpha_1(x)$ never decreases."

(2) On page 332, above the middle of the page, the statement " x approaches c " obviously should read " x approaches the first root of y_1 greater than a ."

(3) Possibly an explicit statement should be made that $V_i(x)$, $i = 1, 2$, does not vanish identically over any interval, as is shown by the conditions on α_1 and α_2 , and that the point l of case I is chosen so that $V_2(l) \neq 0$.

(4) The wording of Theorem I as regards the conditions on K_1 , K_2 , G_1 and G_2 is careless and could be improved by omitting the word "absolutely" and making no statement as to the integrability of K_1 and K_2 .

TOMLINSON FORT.

NOTES.

THE July number (volume 40, number 3) of the *American Journal of Mathematics* contains the following papers: "Interpolation properties of orthogonal sets of solutions of differential equations," by O. D. KELLOGG; "Directed integration," by H. B. PHILLIPS; " P -way determinants, with an application to transvectants," by L. H. RICE; "On a certain general class of functional equations," by W. H. WILSON; "Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order," by R. G. D. RICHARDSON.

THE editors of the *Periodico di Matematica* and its *Supplemento* announce that, on account of the war, the publication of both periodicals will be temporarily discontinued.

AT the meeting of the Edinburgh Mathematical Society held June 14, the following papers were read: By Miss E. PAIRMAN, "Relations connected with generalized differentiation"; by E. T. WHITTAKER, "Some new expansions in series of polynomials."

THE late Professor GASTON DARBOUX left a large part of his library to the newly established reading room of the department of mathematical sciences of the University of Paris.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1918–1919:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Celestial mechanics, three hours.—By Professor L. DONATI: Electrodynamics; electro-optics; relativity, three hours.—By Professor F. ENRIQUES: Foundations of mathematics: (1) geometry, infinitesimal analysis, mechanics and cosmology in ancient Greece; (2) modern critique of principles, three hours.—By Professor S. PINCHERLE: Volterra's and Fréchet's functional calculus; existence theorems for differential equations; integral equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Fourier's series; Dirichlet's problem; spherical and cylindrical functions; functions of a complex variable; elliptic functions; applications, four hours.—By Professor E. DANIELE: Differential equations of mathematical physics with applications, four hours.—By Professor G. SCORZA: Geometry of hyperspaces and some of its applications, four hours.—By Professor C. SEVERINI: Calculus of variations, four and one half hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Infinitesimal geometry of curves and surfaces, three hours.—By Professor O. TEDONE: Partial differential equations with two independent variables and their application to the resolution of physical problems, three hours. By Professor ———: Analysis (advanced course), three hours.

UNIVERSITY OF MESSINA.—By Professor M. BOTTASSO: Vector analysis; newtonian potential; theory of elasticity, three hours.—By Professor P. CALAPSO: Elliptic functions, three hours.—By Professor G. GIAMBELLI: Analytic geometry of hyperspaces; geometrical theory of algebraic elimination, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics (from early times to 1200 A.D.), three hours.—By Professor A. DEL RE: n -dimensional analysis of Grassmann with applications to differential geometry and mechanics, four and one half hours.—By Professor R. MARCOLONGO: Fourier's series; spherical, cylindrical and Lamé's functions; applications, three hours.—By Professor D. MON-

TESANO: Theory of birational transformations of space; involutory birational transformations, three hours.—By Professor E. PASCAL: Selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Geometrical optics, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; integral equations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Electromagnetic field, four hours.—By Professor G. RICCI: Absolute differential calculus with applications, four hours.—By Professor F. SEVERI: Differential geometry, four hours.

UNIVERSITY OF PALERMO: By Professor G. BAGNERA: General analytic functions; entire functions; linear differential equations, three hours.—By Professor M. DE FRANCHIS: Geometry of algebraic surfaces, three hours.—By Professor M. GEBBIA: Mechanics of continuous systems; newtonian and logarithmic potential; hydrostatics and hydrodynamics, four and one half hours.—By Professor A. SIGNORINI: Selected chapters of rational mechanics with particular regard to the theory of elasticity, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Algebraic curves and surfaces, three hours.—By Professor U. CISOTTI: Hydrodynamics, three hours.—By Professor F. GERBALDI: Functions of a complex variable and elliptic functions, three hours.—By Professor G. VIVANTI: Theory of functions of real variables, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry on an algebraic curve, three hours.—By Professor L. BIANCHI: Theory of the continuous groups of transformations, four and one half hours.—By Professor U. DINI: Fourier's series and more general developments concerning the analytic representation of functions of a real variable in a given interval, four and one half hours.—By Professor G. A. MAGGI: Steady and variable electromagnetic fields, four and one half hours.—By Professor ———: Theoretical mechanics (advanced course), three hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor

E. BOMPIANI: Contact transformations of space and their continuous groups, three hours.—By Professor G. CASTELNUOVO: Algebraic equations and groups of substitutions, three hours.—By Professor U. CRUDELI: Theory of continuous groups of transformations, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General wave theory, three hours; Hydrodynamics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Equilibrium forms of a rotating fluid mass, three hours.—By Professor G. FUBINI: Modular, automorphic, fuchsian functions; linear differential equations with rational coefficients, three hours.—By Professor C. SEGRE: Algebraic complexes of straight lines, three hours.—By Professor C. SOMIGLIANA: Thermodynamics; theory of gases; propagation of heat, three hours.

PROFESSORS G. COLONNETTI, of the University of Pisa, E. LAURA, of the University of Pavia, and R. MARCOLONGO, of the University of Naples, has been elected corresponding members of the Reale Istituto Lombardo.

PROFESSOR P. HEEGAARD, of the University of Copenhagen, has been made professor of mathematics at the University of Christiania.

DR. T. BONNESEN has been appointed professor of descriptive geometry at the Copenhagen polytechnic school.

PROFESSOR E. HECKE, of the University of Basel, has been made professor of mathematics at the University of Göttingen.

PROFESSOR H. WEYL, of the polytechnic school of Zurich, has been made professor of mathematics at the University of Breslau.

SIR JOSEPH LARMOR has been awarded the Poncelet prize for the mathematical sciences this year by the Paris Academy of Sciences.

PROFESSOR HORACE LAMB, of the University of Manchester, has been appointed Halley lecturer at Oxford University for the coming year.

AT Clark University Professor A. G. WEBSTER has organized his department of physics for 1918-1919 into a ballistic institute, in which it is proposed to carry on research upon a considerable variety of subjects experimental as well as theoretical.

THE United States Bureau of Education has recently issued a Union List of Mathematical Periodicals prepared by Professor DAVID EUGENE SMITH and Dr. CAROLINE EUSTIS SEELY. This list contains the leading mathematical periodicals needed by research students and to be found in a number of the larger libraries in various parts of the country. Copies may be secured by addressing the United States Commissioner of Education, Washington, D. C.

THE following university and college teachers of mathematics have recently joined the national military service:

Professor ARNOLD DRESDEN, of the University of Wisconsin, Secretary of the Chicago Section of the Society, has gone to France in the service of the Red Cross. Professor W. C. GRAUSTEIN, of Rice Institute, has joined the ordnance at the Aberdeen proving ground. Dr. T. R. HOLLCROFT, of Columbia University, has entered the field artillery officers' training camp at Camp Zachary Taylor, Ky. Professor E. V. HUNTINGTON, of Harvard University, president of the Mathematical Association of America, has been given leave of absence and with the rank of major in the national army is assigned to statistical work under the chief of staff with residence at Washington. Professor P. R. RIDER, of Washington University, has entered the coast artillery training camp at Fort Monroe. Professor J. E. ROWE, of Pennsylvania State College, is engaged in mathematical research for the national advisory committee for aeronautics at Washington. Dr. NORBERT WIENER has joined the ordnance at the Aberdeen proving ground.

PROFESSOR G. A. MILLER, of the University of Illinois, has accepted the chairmanship of a committee which is to make a survey of the mathematical instruction given under the auspices of the Y. M. C. A. at the various naval stations.

PROFESSOR E. D. ROE, JR., of Syracuse University, has been given leave of absence for a year to devote his entire time to research.

At the University of Illinois, the following promotions are announced: associate professor J. B. SHAW to a full professorship, assistant professor R. D. CARMICHAEL to an associate professorship, associates A. J. KEMPNER and G. E. WAHLIN to assistant professorships, instructor E. W. CHITTENDEN to associate, assistant R. F. BORDEN to an instructorship. Associate professor A. B. COBLE, of Johns Hopkins University, has been appointed professor of mathematics. Associate professor HENRY BLUMBERG, of the University of Nebraska, has been appointed associate and Mr. J. B. ROSENBACH, of the University of New Mexico, assistant. Assistant professor C. H. SISAM and Professor H. L. RIETZ have both resigned, Professor SISAM to become head of the department of mathematics at Colorado College and Professor RIETZ to become head of the department at the State University of Iowa.

DR. M. G. GABA, of Cornell University, has been appointed associate professor of mathematics at the University of Nebraska.

ASSISTANT professor W. V. LOVITT, of Purdue University, has been appointed associate professor of mathematics at Colorado College.

DR. FLORA E. LE STOURGEON, of the Liggett School, Detroit, has accepted an instructorship in mathematics at Mount Holyoke College.

DR. I. A. BARNETT has been appointed instructor in mathematics at Washington University.

ASSISTANT professor E. S. SMITH, of the University of Cincinnati, has been appointed acting commandant in addition to his duties in the department of mathematics.

PROFESSOR F. S. NOWLAN, of Brandon College, Brandon, Manitoba, has been appointed instructor in mathematics at Bowdoin College.

MR. J. W. LASLEY, JR., of the University of North Carolina, has been promoted to an assistant professorship of mathematics and given leave of absence for a year to study at the University of Chicago.

PROFESSOR G. CANTOR, of the University of Halle, died January 6, 1918, at the age of 73 years.

PROFESSOR M. SIMON, of the University of Strassburg, died in January, 1918, in his seventy-fourth year.

THE death is announced of Professor E. R. NEOVIUS, of the University of Helsingfors, at the age of 67 years.

THE death is announced of ALBERT GAUTHIER-VILLARS, head of the great publishing house of Paris.

PROFESSOR MAXIME BÔCHER, of Harvard University, died September 12, 1918, at the age of 51 years. Professor Bôcher had been a member of the American Mathematical Society since 1892, and was its president in 1909–1910.

THE Rev. Dr. G. M. SEARLE, superior general of the Paulist Fathers from 1904 to 1909, and previously professor of mathematics and director of the astronomical observatory of the Catholic University at Washington, died on July 8, 1918, at the age of 79 years. Dr. SEARLE had been a member of the American Mathematical Society since 1889.

PROFESSOR A. L. DANIELS, of the University of Vermont, died July 18, 1918, at the age of 69 years. Professor Daniels was made professor emeritus in 1914 after 29 years of service.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

EVANS, G. C. The Cambridge Colloquium, 1916, Part I; Functionals and their applications to selected topics, including integral equations. New York, American Mathematical Society, 1918. 8vo. 12 + 136 pp. Paper. \$1.50; to members of the Society, \$1.00

FORSYTH (A.R.). Functions of a complex variable. 3d edition. Cambridge, University Press, 1918. 24 + 855 pp. 30s.

LANDAU (E.). Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Leipzig, Teubner, 1918. 8vo. 143 pp.

MANFREDI (L.). Geometría proyectiva y descriptiva según los programas de la Facultad de ciencias exactas, físicas y naturales de la Universidad de Buenos Aires. Parte tercera: Aplicaciones. Tercera edición, corregida y notablemente aumentada. (Texto y atlas.) Buenos Aires, Librería científica y casa editora de Augusto Galli, 1917.

OPPERMANN (A.). Note sur le quadrilatère complet. Paris, Gauthier-Villars, 1918. 8vo. 27 pp.

ROGERS (A. L.). Experimental tests of mathematical ability and their prognostic value. (Teachers College, Columbia University, Contributions to Education, No. 89.) New York, 1918. 4 + 118 pp.

SCHOUTEN (G.). De beginselen van de leer der congruentiën. Groningen, P. Noordhoff, 1917. 8vo. 62 pp.

SEELY (C. E.). See SMITH (D. E.).

SETHE (K.). Von Zahlen und Zahlworten bei den alten Aegyptern; ein Beitrag zur Geschichte von Rechenkunst und Sprache. (Schriften der wissenschaftlichen Gesellschaft zu Strassburg, 25tes Heft.) Strassburg, Trübner, 1916.

SMITH (D. E.) and SEELY (C. E.). Union list of mathematical periodicals. (Bureau of Education, Bulletin, 1918, No. 9.) Washington, Government Printing Office, 1918. 8vo. 60 pp.

WIJDENES (P.). Uitgewerkte mondelinge examens hoogere algebra. Groningen, P. Noordhoff, 1917. 8vo. 51 pp. Fl. 5.00

ZEUTHEN (H. G.). Sur la réforme qu'a subie la mathématique de Platon à Euclide, et grâce à laquelle elle est devenue science raisonnée. (Extrait des *Mémoires de l'Académie royale des Sciences et des Lettres de Danemark*: Section des Sciences, 8e série, Tome 1, No. 5.) Copenhagen, Bianco Luno, 1917.

II. ELEMENTARY MATHEMATICS.

BROWN (J. C.). See WENTWORTH (G.).

ELMORE (E. B.). Regents original exercises in plane geometry selected in accordance with the list of propositions adopted by the New York State examinations board. Syracuse, N. Y., C. W. Bardeen, 1918. 8vo. \$0.25

- GONGGRIJP (B.). Theorie der rekenkunde, leerboek voor hogere burger scholen, gymnasia, kweek- en normaalscholen. Eerste stukje. Groningen, P. Noordhoff, 1917. 8vo. 69 pp. Gecart. Fl. 0.65
- GONGGRIJP (B.) en WIJDENES (P.). Antwoorden op de vraagstukken van het leerboek der goniometrie en trigonometrie. Groningen, P. Noordhoff, 1917. 8vo. 38 pp. Fl. 0.60
- Beknopte driehoeksmeting. Groningen, P. Noordhoff, 1917. 8vo. 138 pp. Gecart. Fl. 1.50
- Antwoorden op de vraagstukken van de beknopte driehoeksmeting. Groningen, P. Noordhoff, 1917. 8vo. 29 pp. Fl. 0.50
- HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Second course in algebra. Revised edition. Boston, Ginn, 1918. 8 + 277 pp. Cloth. \$1.00
- LUBY (W. A.). See HAWKES (H. E.).
- SCHLAUCH (W. S.). See WENTWORTH (G.).
- SMITH (D. E.). See WENTWORTH (G.).
- THORNDIKE (E. L.). The Thorndike arithmetics. Books 1-3. New York, Rand and McNally, 1917. 8vo. 16 + 262 + 16 + 288 + 16 + 330 pp. \$0.48 + 0.54 + 0.60
- TOUTON (F. C.). See HAWKES (H. E.).
- WENTWORTH (G.), SMITH (D. E.) and BROWN (J. C.). Junior high school mathematics. Book 3. Boston, Ginn, 1918. 6 + 282 pp. \$0.96
- WENTWORTH (G.), SMITH (D. E.) and SCHLAUCH (W. S.). Commercial algebra. Book 2. Boston, Ginn, 1918. 6 + 250 pp. Cloth. \$1.12
- WIJDENES (P.). See GONGGRIJP (B.).

III. APPLIED MATHEMATICS.

- ASTROGRAPHIC catalogue. Hyderabad section. 1900.0. Vol. 1: Measures of rectangular coordinates and diameters of 63,436 star-images on plates with centres in dec. -170. Edinburgh, Neill, 1918. 43 + 223 pp. 16s. (12 rupees)
- BOILEAU (C.). Le moteur à essence adapté à l'automobile et à l'aviation. Paris, Dunod et Pinat, 1918. 4to. 10 + 179 pp. Fr. 18.00
- BRAHE (T.). Tychonis Brahe Dani opera omnia edidit I. L. E. Dreyer. Tomi quarti fasc. prior. Hauniæ, Libraria Gyldendaliana, 1918. 376 pp.
- BUTLER (G. M.). A manual of geometrical crystallography. New York, Wiley, 1918. 8 + 155 pp. Cloth. \$1.50
- DREYER (I. L. E.). See BRAHE (T.).
- DUMAS (A.). Les aéroplanes au salon de 1913. Paris, Gauthier-Villars, 1914. 8vo. 4 + 82 pp. Fr. 2.75
- DUNCAN (J.) and STARLING (S. D.). A text-book of physics for the use of students of science and engineering. London, Macmillan, 1918. 23 + 1081 pp. 15s.
- ECCLES (J. R.). Lecture notes on light. Cambridge, University Press, 1917. 215 pp. 12s. 6d.
- FRENCH (J.W.). See STEINHEIL (A.).

- GONGGRIJP (B.). Overzicht der mechanica, leerboek voor hogere burgerscholen met 5-jarigen cursus. Groningen, P. Noordhoff, 1917. 8vo. 148 pp. Fl. 2.00
- JAMES (W. H.) and MACKENZIE (M. C.). Principles of mechanism. New York, Wiley, 1918. 244 pp. Cloth. \$1.50
- MACKENZIE (M. C.). See JAMES (W. H.).
- MIALARET (J. H. A.). Handleiding tot de practijk der scheefhoekige en der rechthoekige axonometrie. Amsterdam, L. J. Veen, 1917. 8vo. 136 pp.
- MORLEY (A.). The theory of structures. London, Longmans, 1918. 9 + 584 pp. 14s.
- NACHMAN (H. L.). Elements of machine design. New York, Wiley, 1918. 5 + 245 pp. Cloth. \$2.00
- PEELE (R.). Mining engineers' handbook. New York, Wiley, 1918. 2400 pp. \$5.00
- STARLING (S. D.). See DUNCAN (J.).
- STEINHEIL (A.) and VOIT (E.). Applied optics: the computation of optical systems. Being the "Handbuch der angewandte Optik" of A. Steinheil and E. Voit translated and edited by J. W. French. Vol. 1. London, Blackie, 1918. 17 + 170 pp. 12s. 6d.
- VIDALIE (R.). Etude générale du moteur rotatif. Paris, Dunod et Pinat, 1918. 8vo. 4 + 149 pp. Fr. 7.20
- VILLAMIL (R. DE). Resistance of air. London, Spon, 1918. 10 + 192 pp. 7s. 6d.
- VOIT (E.). See STEINHEIL (A.).
- WIJDENES (P.). Beknopte beschrijven de meetkunde. Groningen, P. Noordhoff, 1917. 8vo. 94 pp. Gecart. Fl. 1.10

The American Mathematical Monthly

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There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual members and over seventy-five institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary at Oberlin, Ohio.

Whole No. 271

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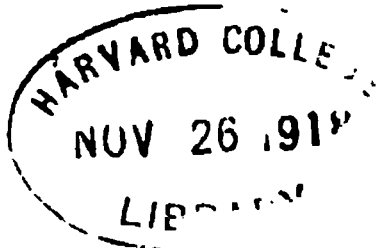
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Articles for insertion in the BULLETIN should be addressed to the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,
501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society :

A. M.	A. M.
Sat., Oct. 26, 1918, 11:00	Sat., Feb. 22, 1919, 11:00
Fri.-Sat., Dec. 27-28, 1918.	Sat., Apr. 26, " "

*The San Francisco Section will meet at the University of California
Saturday, October 26, 1918*

THE TWENTY-FIFTH SUMMER MEETING OF THE
AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth summer meeting of the Society was held, by invitation, at Dartmouth College on Wednesday, Thursday and Friday, September 4-6, 1918, connecting with the meeting of the Mathematical Association, which began on Friday morning. The joint dinner of the two organizations, on Thursday evening, was attended by fifty-six members and friends, who were greeted by Dean Laycock in the name of the College. At the joint session on Friday morning Professor A. G. Webster gave an address on "Mathematics of warfare."

The college dormitories were opened for the accommodation of the visitors, and meals were served in the commons. Headquarters and general gathering place between the sessions was provided in College Hall, where an informal reception was held on Wednesday evening. A letter of welcome from Business Director Keyes tendered the hospitality of the College to the visiting societies. Excursions into the country about Hanover were arranged for the closing days of the meetings. At the joint session a vote of thanks was extended to the college authorities for their generous cooperation toward a successful occasion.

The meeting included an evening session on Wednesday and the usual morning and afternoon sessions on Thursday, as well as the joint session on Friday morning. The attendance included the following forty-six members of the Society:

Professor H. L. Agard, Professor R. C. Archibald, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Daniel Buchanan, Professor W. D. Cairns, Professor W. B. Carver, Professor F. N. Cole, Professor Julia T. Colpitts, Dr. Lennie P. Copeland, Professor Louise D. Cummings, Dr. Mary F. Curtis, Professor C. H. Currier, Professor E. L. Dodd, Mr. T. C. Frye, Professor A. S. Gale, Professor O. E. Glenn, Mr. B. F. Groat, Professor C. F. Gummer, Professor J. G. Hardy, Professor C. N. Haskins, Professor E. V. Huntington, Professor W. W. Johnson, Professor Florence P. Lewis, Professor Helen A. Merrill, Dr. A. L. Miller, Professor F. M. Morgan, Professor G. D. Olds, Professor F. W. Owens, Professor Anna H. Palmié, Professor A. D. Pitcher, Professor J. M. Poor, Mr.

L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. Josephine R. Roe, Professor Clara E. Smith, Professor Sarah E. Smith, Professor H. W. Tyler, Professor Oswald Veblen, Professor C. A. Waldo, Professor A. G. Webster, Mr. J. K. Whittemore, Professor C. B. Williams, Professor F. N. Willson, Professor J. W. Young.

Professor W. W. Johnson presided at the opening session on Wednesday evening, Professor H. W. Tyler at the sessions on Thursday, and Professor G. D. Birkhoff at the joint session on Friday morning. The Council announced the election of the following persons to membership in the Society: Professor A. L. Candy, University of Nebraska; Mr. J. R. Carson, American Telephone and Telegraph Company; Mr. R. S. Hoyt, American Telephone and Telegraph Company; Dr. K. W. Lamson, Columbia University; Professor A. S. Merrill, University of Montana; Mr. F. H. Murray, Harvard University; Mr. H. W. Nichols, Western Electric Company; Professor W. E. Patten, Government Institute of Technology, Shanghai, China. Nine applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. L. B. ROBINSON: "A curious system of polynomials."

(2) Professor G. A. MILLER: "Groups generated by two operators whose relative transforms are equal to each other."

(3) Professor P. J. DANIELL: "Differentiation with respect to a function of limited variation."

(4) Mr. B. F. GROAT: "Models and hydraulic similarity."

(5) Professor L. C. MATHEWSON: "On the groups of isomorphisms of a system of abelian groups of order p^m and type $(n, 1, 1, \dots, 1)$."

(6) Mr. C. N. REYNOLDS: "On the zeros of solutions of linear differential equations of the fourth order."

(7) Professor J. E. ROWE: "Related invariants of two rational sextics."

(8) Professor W. W. JOHNSON: "The nature and history of Napier's rules of circular parts."

(9) Professor O. E. GLENN: "On a new treatment of theorems of finiteness."

(10) Professor LOUISE D. CUMMINGS: "The trains for the 36 groupless triad systems on 15 elements."

(11) Dr. JOSEPHINE R. ROE: "Interfunctional expressibility problems of symmetric functions (third paper)."

- (12) Mr. B. F. GROAT: "Equations of the elastic catenary."
 (13) Professor C. H. FORSYTH: "Relative distributions."
 (14) Professor W. D. CAIRNS: "A derivation of the equation of the normal probability curve."
 (15) Dr. MARY F. CURTIS: "Curves invariant under point transformations of special type."
 (16) Professor G. D. BIRKHOFF: "On stability in dynamics."
 (17) Professor DANIEL BUCHANAN: "Periodic orbits on a surface of revolution."
 (18) Professor C. N. HASKINS: "Note on the roots of the function $P(x)$ associated with the gamma function" (preliminary communication).
 (19) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (third paper)."
 (20) Mrs. CHRISTINE LADD-FRANKLIN: "Bertrand Russell and symbol logic."

Mrs. Ladd-Franklin was introduced by Professor Fiske. In the absence of the authors the papers of Professor Miller, Professor Daniell, Professor Mathewson, Mr. Reynolds, Professor Rowe, Professor Forsyth, and Dr. Schweitzer were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Given the system of equations

$$\begin{aligned}
 p_1 m_{11}^2 + 2p_2 m_{11} m_{21} + 2p_3 m_{11} m_{31} + \\
 p_4 m_{21}^2 + 2p_5 m_{21} m_{31} + p_6 m_{31}^2 = 0, \\
 p_1 m_{11} m_{12} + p_2 (m_{11} m_{22} + m_{12} m_{21}) + p_3 (m_{11} m_{32} + m_{12} m_{31}) \\
 + p_4 m_{21} m_{22} + p_5 (m_{21} m_{32} + m_{22} m_{31}) + p_6 m_{31} m_{32} = 0, \\
 p_1 m_{12}^2 + 2p_2 m_{12} m_{22} + 2p_3 m_{12} m_{32} + p_4 m_{22}^2 \\
 + 2p_5 m_{22} m_{32} + p_6 m_{32}^2 = 0 \quad (p_1 \neq 0);
 \end{aligned}$$

to this system corresponds a determinant $|m_{ij}|$ ($i, j = 1, 2, 3$). Dr. Robinson has demonstrated that, unless the relation

$$(p_2^2 - p_1 p_4) (p_3^2 - p_1 p_6) - (p_2 p_3 - p_1 p_5)^2 = 0$$

exists, the solutions of the given system of polynomials annul all the determinants of the matrix

$$\begin{vmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \end{vmatrix}$$

An analogous theorem holds for systems of polynomials which correspond to the determinant $|m_{ij}|$ ($i, j = 1, 2, \dots, n$).

2. When two operators s and t of a group satisfy the condition $s^{-1}ts = t^{-1}st$, they are of the same order α . If they are not otherwise restricted, Professor Miller shows that they generate a group of order $\alpha(2^\alpha - 1)$ which has a cyclic commutator subgroup of order $2^\alpha - 1$, and is generated by this subgroup and an operator of order α which transforms each of its operators into its square. There is one and only one such group for every positive integral value of α and this infinite system may be called the equitransform system of groups. When $\alpha = 2$ there results the symmetric group of order 6; when $\alpha = 3$, the semi-metacyclic group of order 21; etc. Each group of the system contains a set of $2^\alpha - 1$ conjugate cyclic subgroups of order α and all its operators of order 2 are conjugate whenever such operators occur.

3. In this paper Professor Daniell defines the derivative with respect to a function of limited variation, proves a modification of Vitali's theorem, and finally proves that, under certain restrictions, the original function is the integral of its derivative. A law for integration by parts is found and it is proved that any Stieltjes integral can be expressed as a single Stieltjes integral with respect to a non-decreasing function. The paper is to appear in the *Transactions*.

4. In this paper, Mr. Groat discusses the proper scale ratios to employ to secure similarity of flow of fluids in geometrically similar channels. The determinations are related to theory developed by him in a paper soon to appear in the *Transactions of the American Society of Civil Engineers*, volume 82.

It is shown that similarity of flow requires a scale ratio $L = N^{2/3} \div G^{1/3}$, where L = scale ratio, N = ratio of corresponding kinematic viscosities, and G = ratio of homologous accelerations,—all dimensionless ratios. The similarity then extends to details of the flow, including forces due to viscosity.

Attention is called to the fact that in ordinary model experi-

ments G is unity. Therefore $L = N^{2/3}$. This shows that perfect similarity requires different values of the kinematic viscosity, which in turn requires either different fluids or different viscosity conditions of the same fluid in model and prototype.

Lamb (Hydrodynamics, page 537) states that the kinematic viscosity of air varies inversely as the pressure. If this is true, the sizes of model aeroplane propellers, for example, may be determined for various conditions as follows:

Pressure in atmospheres.....	1000	64	27
Value of N	1/1000	1/64	1/27
Scale ratio of model.....	1/100	1/16	1/9

This assumes, of course, that velocities do not approach too closely the velocity of sound in the air or other fluid involved.

The same effect might be secured by employing another fluid, as mercury, for example. The following table gives approximate ratios in two cases for different conditions: (a) model in water of prototype in air; (b) model in mercury of prototype in water.

Temperature (C.)		Temperature of water (C.)		
		10°	15°	25°
10°	Air.....	.204	.186	.158
	Mercury.....		.219	.255
20°	Air.....	.190	.174	.148
	Mercury.....		.220	.257
40°	Mercury.....		.210	.246

It is of importance to note that the scale ratio of a model in mercury of prototype in air can be obtained by taking the product of any two ratios for an intermediate field. Thus, the scale ratio for air at 20° and mercury at 40° can be secured by taking the product of the air-water and water-mercury ratios for water at any given temperature, say 15°, thus: $L = .174 \times .210 = .036$. Therefore a model aeroplane propeller of about 1/27 full size can be tested in mercury at 40° C., the air being at 20° C., with caution as to the velocity of sound in the media.

5. Moore has shown that the group of isomorphisms of an abelian group of order p^m , and type $(1, 1, \dots, 1)$ is the linear congruence group,* Miller has considered the automorphisms

* Cf. also Burnside, Theory of Groups (1897), §§ 171, 172 and Chap. XIV.

of an abelian group of order p^m , type $(m - 1, 1)$,* and Ranum has developed the group of isomorphisms of any given abelian group of order p^m through a consideration of the group of classes of congruent matrices.† In Professor Mathewson's paper the viewpoint is different and the groups are treated wholly as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of order p^m and type $(n, 1, 1, \dots, 1)$, $n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \dots, 1)$ or that the group of isomorphisms of the system under study ($n = 2, 3, 4, \dots$) is an extension of the one of the system just before it and of index p in it.

6. Professor Birkhoff has proven theorems of oscillation and comparison for the solutions of ordinary linear differential equations of the third order, which are analogous to Sturm's well-known theorems concerning the solutions of linear differential equations of the second order (*Annals of Mathematics*, volume 12 (1911)). In the present paper Mr. Reynolds has proven similar theorems concerning the solutions of self-adjoint linear differential equations of the fourth order, and an oscillation theorem for the general linear differential equation of the fourth order.

7. Professor Rowe's paper appeared in the October BULLETIN.

8. In this paper Professor Johnson gives an historical survey of Napier's rules of the circular parts of right-angled spherical triangles. The principle underlying the rules is examined, and its partial anticipation as found in the early history of trigonometry. Some other cases of circular parts are also discussed.

9. An abstract of Professor Glenn's paper appeared in the March (1918) number of the BULLETIN, in connection with a preliminary report upon the subject, presented by title at the December meeting of the Society. Material since added includes a determination of the complete systems of orthog-

* Miller, *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 259-264.

† Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 71-91.

onal and of boolean invariants and covariants of the binary quintic.

10. In Professor Cummings' paper the sets of trains for the 36 groupless triad systems on fifteen elements are determined, and all types of trains which may occur in the 80 non-congruent systems Δ_{15} are exhibited.

11. In this paper, which is a continuation of those presented at meetings of the Society in October, 1916, and September, 1917, Dr. Roe formulates laws for zero coefficients in interfunctional expressibility tables of symmetric functions and proves triangularity as a property of certain of these tables.

12. Mr. Groat's second paper deals with equations of the elastic catenary deduced about the year 1902. The applicability to suspended cables was established at the time by taking transit observations upon ropes of a cable-way. Since then the theory has been expanded into a treatise adapted to practical engineering, which, though not yet published, has been employed upon transmission-line design and construction.

If Young's modulus be introduced, the following equations result:

$$u + c_0 = \theta + m \sinh \theta,$$

$$v + c_1 = \cosh \theta + \frac{1}{2}m \sinh^2 \theta,$$

$$w + c_2 = \sinh \theta + \frac{1}{2}m (\sinh \theta \cosh \theta + \theta),$$

where $u = x \div c$, $v = y \div c$, $w = z \div c$; u , v , and x , y , being similar systems of rectangular coordinates, and w , z , the corresponding rectifications of the similar curves from vertex to point u , v or x , y . The parameters are m and c , while θ is an independent variable which may be eliminated from any pair of the three equations, leaving a transcendental; for example, the rectangular equation of the elastic catenary.

Moreover, by employing in place of Young's modulus one in which the actual intensity of stress upon the current sectional area replaces that upon the original sectional area of the unstretched cable, another rectangular equation may be derived which is less cumbersome and fully as exact as the former, for all stresses below the elastic limit. There is evidence to show that such a modulus is more nearly constant in actual experiments than Young's.

With certain minor approximations the resulting equations are:

$$mv = \log[1 + m \cosh (\mu u)],$$

$$mw = u - 2 \tanh^{-1} \left[\sqrt{\frac{1-m}{1+m}} \tanh (\mu u) \right],$$

where $\mu = \sqrt{1-m^2}$ and m and c have the same meaning as before, i. e., $m = H \div T$, $c = H \div s$, H being the total tension at the vertex, s the constant weight per unit length of unstretched cable, and T a constant depending upon the size and elastic properties involved.

Thus, it is possible to discuss any elastic catenary by either method as regards the effects of different loadings, temperatures, and other conditions, and this has been accomplished in numerous cases. Practical processes are simplified by employing different parts of the two methods in combination, the results being substantially the same by either, though obtained with greater facility by one or the other.

A general process for elastic curves of all kinds is pointed out, specifically that for an "elastic parabola."

13. In Professor Forsyth's paper a method is derived for making a series of interpolations in a systematic manner in a frequency distribution or a sequence of values separated by equal intervals to give a new frequency distribution or sequence of values separated by equal intervals which may or may not be the same as the original intervals. Thus, for illustration, given the population of a country such as the United States at the middle of the first of each year of several decennial periods, the method may be used to determine the population at the beginning of each such year.

14. The symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. A direct proof, using Stirling's formula, is given by Professor Cairns of the theorem frequently quoted without proof in texts that, as n is increased indefinitely, the polygon (with a finite middle ordinate) approaches as its limiting form the normal curve $y = ke^{-\lambda^2 x^2}$. The method consists essentially in controlling what may be called the points of in-

flexion of the polygons so that these points approach predetermined positions on each side of the mean.

15. In his discussion of the curves defined by a differential equation of the form $dx/X = dy/Y$ Poincaré has investigated the curves invariant under the transformation

$$\begin{aligned} x_1 &= S_1x + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j, \\ \tau: \\ y_1 &= S_2y + \sum_{i+j=2}^{\infty} b_{ij}x^i y^j, \end{aligned}$$

where $S_1 > 1 > S_2 > 0$, $S_2 > 1 > S_1 > 0$; and has shown that there always exist two invariant analytic curves through the invariant point (x_0, y_0) . M. Lattes, using Poincaré's methods, has shown that, when S_1, S_2 are distinct and not zero and one is not a positive integral power of the other, the same result holds. Lie's theory of infinitesimal transformations applied to the case $S_1 = S_2 = 1$ —one of the cases in which Poincaré's methods fail—enables Dr. Curtis to determine the number of curves formally invariant under τ through the origin.

16. Professor Birkhoff's paper is devoted to questions concerning the stability of periodic orbits in dynamical systems with two degrees of freedom. The following theorem forms an important part of this investigation:

Given two closed curves C_1, C_2 about a point O in their plane, each cut by every radial half line from O once and only once, and given a one-to-one, direct, continuous, area-preserving transformation T of the ring C_1C_2 into itself, such that the image of any radial line under T is cut by any radial line at most once; then there exists an infinite connected set of invariant curves C about O , each cut by every radial line once and only once.

The results concerning stability will form part of a paper to appear in the *Proceedings of the American Academy of Arts and Sciences*.

17. In this paper Professor Buchanan determines the periodic orbits described by a particle which moves on a smooth surface of revolution, the axis of which is vertical. Periodic orbits are first determined for the vertical paraboloid of revo-

lution in much the same way as the well-known problem of the spherical pendulum is solved. Then the analytic continuation of these orbits is made with respect to a certain parameter which is introduced to differentiate the more general surface of revolution from the paraboloid.

18. Bourguet showed* that the function

$$P(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \frac{1}{z + \nu}$$

has two real roots in each of the intervals $-2n - 2 < x < -2n - 1$, $n \geq 3$, and that these roots are of the form $-2n - 1 - \xi_n$, $-2n - 2 + \eta_n$, where ξ_n is of the order of $1/(2n)!$. Gronwall has recently shown† that $1/(2n)! < \xi_n < 6/(2n)!$, $1/(2n + 1)! < \eta_n < 6/(2n + 1)!$.

Professor Haskins points out that, though Bourguet's paper contains minor inaccuracies, typographical and other, his approximation is closer than that of Gronwall from $n = 8$ onward and gives the result that $\xi_n < (3 + \epsilon_n)/(2n)!$, where $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Formulas given but not developed by Bourguet lead to an approximation equalling that of Gronwall for $n \geq 2$ and with the limiting value $3/(2n)!$ for large n . Extension of Bourguet's methods leads to still closer approximation, e. g., $2.76/4! < \xi_2 < 2.91/4!$

19. Dr. Schweitzer constructs a table of undefined relations for the postulates for an abstract group, the relations being connoted by a functional notation. The first set of 2^2 undefined relations is of the second degree and has been treated previously.* The k th set of 2^{k+1} undefined relations is of the $(k + 1)$ st degree and is obtained by induction from the set for $k = k_0$; e. g., for $k = 3$ the indefinables are, $F_{13}(x_1x_2x_3)$, $F_{23}(x_1x_2^{-1}x_3)$, $F_{33}(x_1^{-1}x_2x_3)$, $F_{43}(x_1^{-1}x_2^{-1}x_3)$, $F_{53}(x_1x_2x_3^{-1})$, etc. Conceivably in the case of k variables one has $\sum_{s=1}^m {}^m C_s$ classes of systems of postulates for an abstract group, where the sub-

* Bourguet, L., *Comptes Rendus*, vol. 96 (1883), pp. 1307-1310; 1457-1490.

† Gronwall, T. H., *Annales Scientifiques de l'Ecole Normale Supérieure*, series (3), vol. 33 (1916), pp. 381-393.

* Compare abstracts, BULLETIN, May, 1918, p. 371, June, 1918, p. 428. In the former abstract, the relations II_2 , II_3 , II_1 , should be stated as follows: II_2 . $\phi\{f(x, y), y\} = x$, II_3 . $f\{\phi(x, y), y\} = x$, II_1 . $f\{f(y, x), f(z, x)\} = f(y, z)$.

script s denotes the number of indefinables involved in a system of postulates and $m = 2^{k+1}$.

In the present paper the author constructs for $k = 2$, five classes of systems of postulates for a group, finite or infinite, based on three and four indefinables respectively; also eight systems of postulates are constructed for $k = 3$, each system involving a single indefinable, F_{13} , . . . , F_{83} respectively. Infinitudes of new equations in iterative compositions are given (with special reference to the preceding 2^{k+1} indefinables) and their interpolation as functional equations is investigated. These iterative relations are found in partial solution of the problem: To find all equations in iterative compositions of order $w = 1, 2, 3, \dots$, satisfied by the elements of an abstract group, finite or infinite.

20. Dr. Ladd-Franklin maintains that one reason why the great work of Bertrand Russell—the reduction of all mathematics to a few pure-logic “primitives” (the word here proposed to cover at once undefinable terms and undemonstrable propositions)—has not been more widely accepted than it has is because his views are, hitherto, somewhat subject to change. His first book, the *Foundations of Geometry*, has long since been substantially discarded. Few who read the *Principles of Mathematics* (1903) realize the extent to which it has already been superseded by the *Principia Mathematica* (1910). In this interval it has been found necessary to give up not only the whole doctrine of classes but also that of propositional functions and even of relations in general. Insurmountable difficulties which presented themselves have been solved, at present, by the drastic “no classes theory,” though it was thought at first (see *Proceedings of the London Mathematical Society*, 1905, for the very illuminating discussion) that the zigzaggedness theory or the small classes theory might serve the purpose better. One naturally waits a bit longer to see if the “no classes theory” may not also prove to be fallible.

A defect that certainly still remains in the doctrine of Peano and Russell—an example of the several infelicities of their form of symbol logic—is the so-called new relation “epsilon”; there is nothing peculiar in the *relation* concerned—the specificity lies simply in the subject term, which is “individual” or singular. The only reason they give for regarding the relation as peculiar—that it is not subject (as is the relation of

implication in general) to the rule of syllogism—is wholly fallacious; the real source of the difference lies in the change of sense of the middle term; that to confound in a middle term the *sensus compositi* and the *sensus divisi* is a source of danger has been a commonplace of logic since the time of the scholastics. That an inept symbolism is made use of in mathematics, which has for a fundamental interest the point and the “variable” (i. e., individuals, or singulars) would be of no consequence, but Russell and Peano treat this “addition” as constituting an important improvement over the logic which preceded them—that of Peirce and his school—instead of which it is simply erroneous.

F. N. COLE,
Secretary.

ON THE HEINE-BOREL PROPERTY IN THE THEORY OF ABSTRACT SETS.

BY DR. E. W. CHITTENDEN.

(Read before the American Mathematical Society, October 26, 1918.)

O. VEBLEN and N. J. Lennes have shown that in the presence of certain linear order axioms the Heine-Borel property is equivalent to the Dedekind cut axiom.* O. Veblen† and R. L. Moore‡ have used this property in systems of axioms for geometry and analysis situs.

M. Fréchet established the theorem that in a class (V) normale a subclass Ω has the Heine-Borel property if and only if Ω is compact and closed.§ This result was extended to systems $(\Omega; K^{1867})$ by T. H. Hildebrandt.||

E. R. Hedrick calls attention to the fact that the Heine-Borel theorem, in the enumerable case, is a consequence of the closure of derived classes.¶

* Cf. O. Veblen, “The Heine-Borel theorem,” this BULLETIN, vol. 10 (1904), p. 436–439.

† “A system of axioms for geometry,” *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 343–384.

‡ “On the foundations of plane analysis situs,” *ibid.*, vol. 17 (1916), p. 131.

§ “Sur quelques points du calcul fonctionnel,” *Rendiconti di Palermo*, vol. 22 (1906), p. 26.

|| “A contribution to the foundations of Fréchet’s calcul fonctionnel,” *Amer. Journal of Math.*, vol. 34 (1912), pp. 281–282.

¶ “On properties of a domain for which the derived set of any set is closed,” *Transactions Amer. Math. Society*, vol. 12 (1911), pp. 285–294.

In a previous paper* I have shown that the Heine-Borel property implies self-compactness and extremality in any set $\mathfrak{R}$ of Fréchet. The present paper considers the Heine-Borel property in relation to the closure of derived classes and the property of separability. The converse of the theorem of Hedrick is established.

1. Let $\mathfrak{R}$ denote a fundamental set of elements called points such that for each point P and subset $\mathfrak{P}$ it is determined whether or not P is a limiting point of $\mathfrak{P}$. F. Riesz† regards the following four axioms as fundamental in the theory of systems $\mathfrak{R}$:

(A) If a point P is a limiting point of a set $\mathfrak{P}$ and $\mathfrak{P}$ is a subset of $\mathfrak{Q}$, then P is a limiting point of $\mathfrak{Q}$.

(B) If $\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2$ and P is a limiting point of $\mathfrak{P}$, then P is a limiting point of $\mathfrak{P}_1$ or $\mathfrak{P}_2$.

(C) If P is a limiting point of $\mathfrak{P}$, then $\mathfrak{P}$ contains at least two elements.‡

(D) A point P is determined uniquely by the sets of which it is a limiting point.

A point P is *interior* to a set $\mathfrak{S}$ if $\mathfrak{S}$ contains P and a point of every set which has P for a limiting point.§

A family ($\mathfrak{S}$) of sets $\mathfrak{S}$ of points *covers* a set $\mathfrak{Q}$ if every point of $\mathfrak{Q}$ is *interior* to some class $\mathfrak{S}$.

A class $\mathfrak{Q}$ has the Heine-Borel property if every family ($\mathfrak{S}$) which *covers* $\mathfrak{Q}$ contains a finite subfamily

$$\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \dots, \mathfrak{S}_n,$$

which also covers $\mathfrak{Q}$. The enumerable case of this property is defined by the restriction "if every *enumerable* family ($\mathfrak{S}$), etc." When the families ($\mathfrak{S}$) are not subject to this limitation I shall speak of the general case.

THEOREM 1. If $\mathfrak{R}$ satisfies axiom (A) and $\mathfrak{P}$ is a subset of $\mathfrak{R}$ possessing the Heine-Borel property, then any closed subset of $\mathfrak{P}$ has the same property.

* "On the converse of the Heine-Borel theorem in a Riesz domain," this BULLETIN, vol. 21 (1915), pp. 17-183. Cf. Theorems I and IV.

† Cf. "Stetigkeitsbegriff und abstrakte Mengenlehre," Atti del IV Congresso Inter. dei Mat., Roma (1908), vol. 2, pp. 18-24.

‡ From axioms (B) (C) it follows that, if $\mathfrak{P}$ has a limiting point, $\mathfrak{P}$ is an infinite class.

§ In the previous paper I assumed that this point be distinct from P . R. L. Root called my attention to the fact that this condition is unnecessary in view of axioms (B) and (C). The removal of the restriction permits a generalization of Theorem I of the previous paper, also suggested by Root.

Let Ω be a closed subset of $\mathfrak{P}$ and $(\mathfrak{S})$ be a family of sets covering Ω . Denote by $\mathfrak{S}$ the class of all points of $\mathfrak{R}$ not contained in Ω . From the closure of Ω and axiom (A) it follows that no subset of Ω has a limiting point in $\mathfrak{S}$. The family of sets obtained by adding $\mathfrak{S}$ to $(\mathfrak{S})$ covers $\mathfrak{P}$, and must contain a finite subfamily with the same property. This finite subfamily contains $\mathfrak{S}$, but $\mathfrak{S}$ contains no point of Ω . Hence $(\mathfrak{S})$ contains a finite subfamily which covers Ω , as was to be proved.

The following theorem is a generalization of theorem 1 of the previous paper, suggested to me by R. E. Root. As the proof of the present theorem is only slightly different from the proof of the original theorem, it is omitted here.

THEOREM 2. *If $\mathfrak{R}$ satisfies axiom (A) and $\mathfrak{P}$ is any subset of $\mathfrak{R}$ possessing the Heine-Borel property in the enumerable (or general case), then $\mathfrak{P}$ is self-compact; that is, every infinite subclass of $\mathfrak{P}$ has a limiting point in $\mathfrak{P}$.*

Theorems 1 and 2 are easily seen to hold for the general sets $(\mathfrak{B})$ recently introduced by Fréchet.*

2. It follows from theorem 2 that if $\mathfrak{R}$ satisfies axiom (A) and contains an infinite class with the Heine-Borel property then there is an element in $\mathfrak{R}$ which is a limiting point of an enumerable set. The following example shows that not all points of $\mathfrak{R}$ need have this property. Let $\mathfrak{R}$ denote the closed interval $(0, 1)$ with the additional point $P = 2$. Limiting point is defined as usual for the interval $(0, 1)$. The point $P = 2$, however, is to be a limiting point of every subset of $\mathfrak{R}$ which has the power of the continuum. Then $\mathfrak{R}$ satisfies axioms (A), (B), (C), (D), and also has the Heine-Borel property. Theorem 2 is valid, but the point $P = 2$ is not a limiting point of any enumerable set. The derived class of the rational points of $\mathfrak{R}$ is the interval $(0, 1)$, which is not closed in $\mathfrak{R}$. Hence closure of derived classes is not implied by the Heine-Borel property in the set $\mathfrak{R}$ as conditioned by Riesz. If we assume in place of axiom (D) the stronger axiom:

(D') If P is a limiting point of a set $\mathfrak{P}$, then $\mathfrak{P}$ contains an enumerable set $\mathfrak{S}$, of which P is the only limiting point; we shall obtain a system $\mathfrak{R}$ in which closure of derived classes is implied by the Heine-Borel property. We have

THEOREM 3. *If $\mathfrak{R}$ satisfies axioms (A), (B), (C), (D') and any class Ω in $\mathfrak{R}$ possesses the Heine-Borel property in the enumerable case, the derived class of every subclass of Ω is closed.*

* *Comptes Rendus*, vol. 165 (1917), pp. 359-60.

We will first prove that Ω is closed. From axiom (D') any limiting point Q of $\mathfrak{K}$ is a unique limiting point of some enumerable subset $\mathfrak{S}$ of Ω . $\mathfrak{S}$ is an infinite set (axioms (A) , (B) , (C)) and from theorem 2 has a limiting point in $\mathfrak{S}$. We shall now show that a contradiction results if we assume that Ω contains a set whose derived set is not closed. Let $\mathfrak{D}$ be a subset of Ω such that $\mathfrak{D}'$ has a limiting point Q not in $\mathfrak{D}'$. Then there is a sequence of points of $\mathfrak{D}'$,

$$\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \dots$$

with the unique limiting point Q . Denote by $\mathfrak{S}_n$ the class of all points of $\mathfrak{K}$ excepting the points $\mathfrak{D}_{n+h}$ ($h = 1, 2, 3, \dots$), and $\mathfrak{D}$. It is evident that each point of Ω is interior to at least one of the classes $\mathfrak{S}_n$. Denote by $\mathfrak{S}$ the set of all points not contained in the set $\mathfrak{D}$. Then the point Q is interior to $\mathfrak{S}$. The family

$$\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_2, \dots$$

is enumerable and covers the set Ω . It contains a finite subfamily which also covers Ω and of which $\mathfrak{S}$ is a member. No class $\mathfrak{S}_n$ contains more than n of the points O_n . Therefore some point O_{n_0} must be interior to $\mathfrak{S}$. Since the point O_{n_0} is an element of the set $\mathfrak{D}'$, some point of $\mathfrak{D}$ must belong to $\mathfrak{S}$. This is the desired contradiction.

In a class $\mathfrak{L}$ of Fréchet* a point P is a limiting point of a class $\mathfrak{P}$ if and only if $\mathfrak{P}$ contains a sequence of distinct elements with the limit P . The class $\mathfrak{L}$ is a class $\mathfrak{K}$ satisfying axioms (A) , (B) , (C) , (D') . From theorem 3 and a theorem of E. R. Hedrick† we have at once:

THEOREM 4. *In any set $\mathfrak{L}$ of Fréchet a necessary and sufficient condition that a subset $\mathfrak{P}$ of $\mathfrak{L}$ possess the Heine-Borel property is that $\mathfrak{P}$ be compact, closed, and that the derived set of every subset of $\mathfrak{P}$ be closed.*

3. F. Hausdorff‡ employs a form of the Heine-Borel property which does not imply the closure of derived sets in every system $\mathfrak{L}$ of Fréchet. The Hausdorff form of this property may be defined as follows: If a class $\mathfrak{P}$ is covered by a family $(\mathfrak{G})$ of regions§ $(\mathfrak{G})$ then $\mathfrak{P}$ is covered by a finite subset of $\mathfrak{G}$.

* *Rendiconti di Palermo*, loc. cit., p. 1, 2.

† Loc. cit., p. 286.

‡ *Grundzüge der Mengenlehre*, Veit and Co., Leipzig (1914), p. 231.

§ A region (Gebiet) is a set whose every element is an interior element. Cf. Hausdorff, loc. cit., p. 215, footnote 1.

Let the class $\mathfrak{I}$ be the interval $(0, 1)$. We define limit as usual in $(0, 1)$ with the single exception of the sequence of points $x = 1/m$ ($m = 2, 3, \dots$). The sequence $\Sigma(1/m)$ is to have the unique limit $x = 1$. Any subset $\mathfrak{P}$ of $\mathfrak{I}$ is compact, but derived classes are not closed. Let $\mathfrak{D}$ denote the irrational points of the interval $(0, 1/2)$. Then $\mathfrak{D}'$ is the interval $(0, 1/2)$, while $\mathfrak{D}''$ contains $x = 1$. Nevertheless it is easy to see that any closed subset of $\mathfrak{I}$ has the Hausdorff form of the Heine-Borel property. We give an example of a family $(\mathfrak{S})$ which covers $\mathfrak{I}$ and cannot be replaced by a finite subfamily.

Let $\mathfrak{S}_n$ be the class of all points of $(0, 1)$ except the points $x = 1$ and $x = 1/m$ ($m > n$). Denote by $\mathfrak{S}_0$ the class of all points not an irrational point of the interval $(0, 1/2)$. Then

$$\mathfrak{S}_0, \mathfrak{S}_1, \dots, \mathfrak{S}_n, \dots$$

is an enumerable family which covers $\mathfrak{P}$. No class $\mathfrak{S}_n$ contains more than a finite number of the points $x = 1/m$, and no point $x = 1/m$ ($m > 1$) is interior to $\mathfrak{S}_0$. It follows that the family $(\mathfrak{S})$ cannot be replaced by a finite subfamily. The set $\mathfrak{S}_0$ is not a region.

4. Fréchet* has studied classes $(\mathfrak{B})$ in which for each element there is given a family $(\mathfrak{V})$ of neighborhoods (voisinages). We shall impose the following conditions on the families $(\mathfrak{B})$: (1) For every element (point) P the corresponding family $(\mathfrak{V})$ is enumerable and is denoted by $(\mathfrak{V}^m)$; (2) the point P is contained in every neighborhood $\mathfrak{V}^m$ of P and is the only such point: (3) in a family $(\mathfrak{V}^m)$, $\mathfrak{V}^m$ contains $\mathfrak{V}^{m+1}$; (4) if a point Q is in a neighborhood $\mathfrak{V}^m$ of P then P is in the corresponding neighborhood $\mathfrak{V}^m$ of Q .†

A point P is a limiting point of a class $\mathfrak{P}$ if every neighborhood of P contains a point of $\mathfrak{P}$ distinct from P .

THEOREM 5. *If $\mathfrak{Q}$ is a set in a system $(\mathfrak{B})$ with the Heine-Borel property, and if $\mathfrak{Q}'$ contains $\mathfrak{Q}$, then $\mathfrak{Q}$ is compact, perfect, and separable.*

Since $\mathfrak{Q}'$ contains $\mathfrak{Q}$ and $\mathfrak{Q}$ is compact (theorem 2) it follows that $\mathfrak{Q}$ is perfect. We have to prove that $\mathfrak{Q}$ contains an enumerable subset $\mathfrak{E}$ such that $\mathfrak{Q} = \mathfrak{E}'$.

Each point P is interior to the corresponding neighborhood

* *Comptes Rendus*, loc. cit.

† The class $(\mathfrak{B})$ admits a definition of a symmetric distance function δ , such that $\delta(P, Q) = 0$ if and only if $P = Q$.

$\mathfrak{B}^m$. The system of neighborhoods $\mathfrak{B}^m$ for fixed m covers Ω and may be replaced by a finite subsystem,

$$\mathfrak{B}^{m1}, \quad \mathfrak{B}^{m2}, \quad \dots \quad \mathfrak{B}^{mk};$$

such that each point of Ω is interior to some class $\mathfrak{B}^{mk}$ and each class $\mathfrak{B}^{mk}$ is a neighborhood of a point Q^{mk} of the class Ω . Let $\mathfrak{C}$ be the class of all points Q^{mk} . Since every point P of Ω is interior to some set $\mathfrak{B}^{mk}$, it follows from condition (4) that Q^{mk} is contained in the neighborhood $\mathfrak{B}^m$ of P . Therefore P is a limiting point of the class $\mathfrak{C}$.

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This article was in type before the writer learned of the existence of an article by Fréchet (*Bulletin de la Société mathématique de France*, volume 35, 1917), in which it is shown that the *closure of derived classes* is a consequence of the Heine-Borel property in the case of a general system $\mathfrak{L}$. Theorem 3 of the present paper is a generalization of this result.

E. W. CHITTENDEN.

INTEGRALS AROUND GENERAL BOUNDARIES.

BY PROFESSOR P. J. DANIELL.

THE concept of a boundary integral has been extended to curves of the type $x = x(t)$, $y = y(t)$, where $x(t)$, $y(t)$ are absolutely continuous functions of a parameter t . In this case the curves are more or less simple and have tangents "nearly everywhere." In applications to physics however the boundary must be considered rather as a boundary of a set (in the sense of the theory of point sets). The boundary will be, in general, a collection of points without definite tangents at all. This paper sets out a method by which such boundary integrals can be defined under certain restrictions placed on the two integrand functions u , v . The method depends on the concept of absolutely additive functions of sets.* The writer believes that these restrictions could be lightened and that there is a wide field here for further investigation.

* J. Radon, *Wiener Sitzungsberichte*, vol. 122 (1913), p. 1295. W. H. Ycung, *Proceedings London Math. Society*, vol. 13 (1914), p. 109.

Statement of Problem.—Given any set E measurable Borel, and its boundary $B(E)$, contained in a closed fundamental interval J ($0 \leq x \leq 1, 0 \leq y \leq 1$); given also two functions $u(x, y), v(x, y)$ summable in the interval J ; to define

$$\int_{B(E)} u dx + v dy.$$

Note.—The boundary integral is taken in such a sense that on a rectangle for the side with the lesser value of y the integral is taken in the direction of x increasing. As the axes are usually drawn this corresponds to a counter-clockwise sense.

Definitions and Restrictions.—

- R 1. Let the total variation of $u(x, y)$, varying y , be $\lambda(x)$, where $\lambda(x)$ is finite nearly everywhere in x and summable in $(0 \leq x \leq 1)$.
- R 2. Let the total variation of $u(x, y)$, varying x , be $\mu(y)$, where $\mu(y)$ is finite nearly everywhere in y and summable in $(0 \leq y \leq 1)$.

We shall consider in the first place rectangles r with sides parallel to the axes. Then

$$\int_{B(r)} u dx + v dy = \int_r d\alpha(x, y)$$

can be proved to be an absolutely additive function of rectangles r , and we may define

$$\int_{B(E)} u dx + v dy = \int_E d\alpha(x, y).$$

If $d\alpha(x, y)$ is an absolutely additive function of rectangles we can by Radon's method define $\int_E d\alpha(x, y)$ uniquely for any set E measurable Borel contained in J . All that is needed then is to prove that $\int_{B(r)} u dx + v dy$ is an absolutely additive function of rectangles r and to define

$$\alpha(x, y) = \int_{B(r')} u d\xi + v d\eta,$$

where r' is the rectangle $(0 < \xi < x, 0 < \eta < y)$.

Proof. By R_1 , the total variation of $u(x, y)$ in $(0 \leq y \leq 1)$

is $\lambda(x)$. Denote the total variation in $(0 \leq \eta \leq y)$ by $\lambda(x)\theta(x, y)$ when $\lambda(x)$ is finite. $\lambda(x)$ is non-negative, $\theta(x, y)$ is non-negative and a non-decreasing function of y taking the value 0 when $y = 0$, and 1 when $y = 1$. In particular it is a limited measurable function of x . Define

$$\sigma_1(x, y) = \int_0^x \lambda(x)\theta(x, y)dx.$$

$$\begin{aligned} \int_r d\sigma_1(x, y) &= \sigma_1(x_1, y_1) + \sigma_1(x_2, y_2) - \sigma_1(x_1, y_2) - \sigma_1(x_2, y_1) \\ &= \int_{x_1}^{x_2} \lambda(x)dx[\theta(x, y_2) - \theta(x, y_1)] \end{aligned}$$

is a finite non-negative additive function of rectangles. Then for any set E measurable Borel

$$\int_E d\sigma_1(x, y) \text{ is defined and } \leq \int_J d\sigma_1(x, y) \text{ or } S_1.$$

By R_1 ,

$$|u(x, y_1) - u(x, y_2)| \leq \lambda(x)[\theta(x, y_2) - \theta(x, y_1)].$$

$u(x, y)$ is summable in (x, y) or is summable in x for nearly all values of y . Let y_0 be one of the values for which it is summable. Then

$$|u(x, y)| \leq |u(x, y_0)| + \lambda(x)|[\theta(x, y) - \theta(x, y_0)]|$$

or $u(x, y)$ is summable in x for *all* values of y .

$$\begin{aligned} \left| \int_{B(r)} u dx \right| &= \left| \int_{x_1}^{x_2} [u(x, y_1) - u(x, y_2)] dx \right| \\ &\leq \int_{x_1}^{x_2} \lambda(x)[\theta(x, y_2) - \theta(x, y_1)] \end{aligned}$$

or

$$\int_r d\sigma_1(x, y).$$

Hence for any set $\sum_{i=1}^n r_i$ of non-overlapping rectangles

$$\begin{aligned} \sum_{i=1}^n \left| \int_{B(r_i)} u dx \right| &\leq \int_{\sum_{i=1}^n r_i} d\sigma_1(x, y) \\ &\leq S_1. \end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty} \int_{B(r_i)} u dx$$

is absolutely convergent.

Moreover $\int_{B(r)} u dx$ is an additive function of rectangles r , for if two or more rectangles have some parts of their boundaries in common (but do not overlap), the integrals along these parts being taken in opposite directions will annul each other.

If we define

$$\alpha_1(x, y) = \int_{B(r')} u dx, \quad r' = (0 \text{ to } x, 0 \text{ to } y),$$

$$\int_{B(r)} u dx = \int_r d\alpha_1(x, y)$$

defines an absolutely additive function of rectangles. Similarly for

$$\int_{B(r)} v dy = \int_r d\alpha_2(x, y)$$

and therefore also for

$$\int_{B(r)} u dx + v dy = \int_r d\alpha(x, y),$$

where $\alpha(x, y) = \alpha_1(x, y) + \alpha_2(x, y)$.

This was to be proved, and it follows that we can define

$$\int_{B(E)} u dx + v dy = \int_E d\alpha(x, y).$$

More generally, the same method could be used if it can be proved that $\int_{B(r)} u dx + v dy$ is an absolutely additive function of rectangles; the difficulty is to state the required conditions as conditions on u and v directly. That is the reason for the introduction of R_1 , R_2 , which are sufficient but probably not necessary.

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DETERMINANT GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society April 13, 1918.)

§1. *Introduction.*

LET D represent a determinant of order n whose n^2 elements are regarded as independent variables. The substitutions on these n^2 elements which transform D into itself constitute a substitution group G , which we shall call the *determinant group* of degree n^2 . As the elements of D are supposed to be independent variables, it results from the definition of a determinant that every substitution of G must transform the elements of D in such a manner that all the elements of a line (row or column) appear in a line after the transformation.

Hence the substitutions of G correspond to the permutations of the elements of D resulting from transforming its rows and columns independently according to the alternating group of degree n , transforming its rows and columns simultaneously according to negative substitutions in the symmetric group of this degree, and interchanging the rows and columns. The order of G is therefore $(n!)^2$, and hence the number of the distinct determinants that can be formed by permuting the n^2 elements of D is $n^2!/(n!)^2$.* These determinants may be arranged in pairs such that each pair is composed of the determinants which differ only with respect to sign. In particular, the square of D is transformed into itself by a group K whose order is twice the order of G and which contains G as an invariant subgroup.

Some of the abstract properties of G follow directly from the fact that it is simply isomorphic with the imprimitive substitution group of degree $2n$ whose head is composed of the positive substitutions in the direct product of two symmetric groups of degree n . These substitutions correspond to interchanges of the rows among themselves and the columns among themselves or a combination of such interchanges. The re-

* G. Bagnara, *Giornale di Matematiche*, vol. 25 (1887), p. 228; it may be noted that in the review of this article in *Jahrbuch über die Fortschritte der Mathematik* the author's name appears in the form Bergnera.

maining substitutions of this imprimitive group correspond to the interchanges of rows and columns. Hence it results directly that *G contains exactly three subgroups of index 2 for every value of n .*

When $n > 4$, G contains no invariant subgroup besides the three of index 2 mentioned above and their common subgroup of index 4 under G , which is also the commutator subgroup of G . In this case G is unsolvable and its factors of composition are 2, 2, $n!/2$, $n!/2$. When $n = 2$, G is simply isomorphic with the four-group, when $n = 3$ it is simply isomorphic with the square of the non-cyclic group of order 6, and when $n = 4$ it contains an invariant abelian subgroup of order 16 and of type (1, 1, 1, 1) which is complementary to a quotient group of order 36 simply isomorphic with the square of the non-cyclic group of order 6.

§ 2. *Determinant Group as a Substitution Group of Degree n^2 .*

It has been noted that the substitution group of degree $2n$ which is simply isomorphic with G and corresponds to the permutation of the rows and columns of D is always imprimitive. On the other hand, it is easy to prove that G itself is intransitive when $n = 2$, imprimitive when $n = 3$, and both simply transitive and primitive for every value of n which exceeds 3. In fact, the cases when $n = 2$ or 3 can readily be established by means of the well-known lists of possible substitution groups of degrees 4 and 9.*

That G is always simply transitive when $n > 3$ results directly from the fact that all its substitutions which omit a given letter constitute a subgroup having two transitive constituents. One of these is on the $2n - 2$ other letters in the row and column which contain the fixed letter, while the other transitive constituent is on the $(n - 1)^2$ letters not found in this row or this column. The fact that G is primitive is a consequence of the theorem that *if the subgroup composed of all the substitutions which omit one letter of an imprimitive group omits only one letter, then the degree of one of the transitive constituents of this subgroup increased by one must divide the degree of the imprimitive group.*

The group G contains exactly two substitutions which trans-

* F. N. Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 252.

form each of the elements of any term of D into itself. In particular, the principal term of D is transformed into itself by the substitution of order 2 which interchanges the k th row of D and the k th column, $k = 1, 2, \dots, n$. Each of the terms of D is therefore transformed into itself by at most $2 \cdot n!$ of the substitutions of G . As such a term cannot be transformed under G into more than $n!/2$ terms, it results that this is the actual number of the substitutions of G which transform a term into itself. The elements of each term of D must therefore be transformed under G according to the symmetric group of degree n .

The positive terms of D , as well as the negative ones, are transformed under G according to a simply transitive group of degree $n!/2$ since the order of G is not divisible by $n!/2 - 1$, whenever $n > 3$. The two substitutions of G which transform each of the elements of one of these terms into itself constitute a group of degree $n(n - 1)$, hence G involves two complete sets of $n!/2$ conjugate substitutions of order 2 and of degree $n(n - 1)$. These sets form a single set of conjugates under the group of D^2 and each of their substitutions is invariant under a group which is simply isomorphic with the direct product of the symmetric group of degree n and a group of order 2. As this group of order $2 \cdot n!$ is a maximal subgroup of G , it results that the simply transitive groups of degree $n!/2$ according to which the positive and the negative terms of D are transformed under G must be primitive whenever $n > 3$.

It was noted above that the subgroup composed of all the substitutions of G which omit a given letter has two transitive constituents of degrees $2(n - 1)$ and $(n - 1)^2$ respectively. This subgroup is evidently formed by a simple isomorphism between these constituent groups and hence each of these groups has for its order the square of $(n - 1)!$. The former constituent is imprimitive, being formed by extending, by a substitution of order 2 which merely interchanges the corresponding letters, the positive substitutions in the direct product of two symmetric groups of degree $n - 1$ written on distinct sets of letters. The latter constituent is the group of the determinant of order $n - 1$ and hence is a simply transitive primitive group, whenever $n > 4$. In particular, *the subgroup of the determinant group of degree n^2 which is composed of all its substitutions omitting a given letter is simply isomorphic with the group of the determinant of order $n - 1$ and has this group for a transitive constituent whenever $n > 3$.*

Although G is a simply transitive primitive group whenever $n > 3$, it always contains a subgroup of index 2 which is imprimitive, viz., the subgroup corresponding to the interchanges of the rows and columns according to the positive substitutions in the square of the symmetric group. In fact, this subgroup has two sets of systems of imprimitivity, one composed of the elements of rows and the other composed of elements of columns. Each system of the first set has one and only one element in common with each system of the second set. This subgroup contains no other set of systems of imprimitivity and these two sets are transformed into each other under the primitive group G .

The simply transitive primitive group G is evidently of class $3n$, $n > 3$, and hence we have here an interesting infinite system of simply transitive primitive groups in which there are groups for which the ratio of the degree to the class exceeds any given finite number, this ratio being $n/3$. In § 4 we shall consider another infinite system of such groups for which this ratio is still larger. When a primitive group is at least doubly transitive it is well known that this ratio cannot exceed 4 unless the group is either alternating or symmetric.*

§ 3. *Determinant Groups as Substitution Groups of Degree $n!/2$.*

When $n = 2$, D has only one positive term, which is therefore invariant under G . When $n = 3$, D has three positive terms, which are transformed under G according to the symmetric group of degree 3 and therefore each of these three terms is transformed into itself by six of the substitutions of G . These six substitutions transform the three negative terms of D according to the symmetric group of degree 3, hence G contains no substitution besides identity which is commutative with each of the six terms of D .

It has been noted that G transforms the $n!/2$ positive terms of D according to a simply transitive primitive group G' , which is simply isomorphic with G whenever $n > 3$ because G involves no invariant subgroup whose index exceeds 4. Let T_1 be the principal diagonal term of D , and let T_2 be a positive term of D which has the last $n - 3$ elements in common with T_1 but is not identical with T_1 . Among the $2 \cdot n!$ substitutions of G which transform T_1 into itself there are

* Cf. Encyclopédie des Sciences mathématiques, Tribune publique 14, no. 299.

$6 \cdot (n - 3)!$ which also transform T_2 into itself. The subgroup of G composed of all its substitutions which omit a given letter must therefore contain a transitive constituent of degree $[n(n - 1)(n - 2)]/3$.

The number of the transitive constituents of this subgroup varies with n and increases without limit as n increases without limit, since the positive terms which have α elements in common with T_1 can clearly not be transformed under this subgroup into those which have $\beta \neq \alpha$ elements in common with T_1 . When $n > 4$ each of these constituents is either of order $n!$ or of twice this order, because this subgroup has only one invariant subgroup whose index exceeds 4 and could not have a transitive constituents of order 2 or of order 4. In fact, each term which has an element in common with T_1 has evidently more than four conjugates under this subgroup whenever $n > 4$, and if a term has no element in common with T_1 its element which occurs in the first row, for instance, can be transformed into $n - 1$ other elements of this row under this subgroup.

Among the substitutions which transform T_1 into itself there are $2^{e+1} \cdot e!$ substitutions which also transform the secondary diagonal term of D into itself, e being the largest integer which does not exceed $n/2$. Hence it results that the secondary diagonal term of D is always transformed into an odd number of conjugates under the group formed by the substitutions which transform T_1 into itself, $2 \cdot n! \div 2^{e+1} \cdot e!$ being an odd integer as can readily be proved. In particular, when the secondary diagonal term of D is positive the subgroup of G' composed of all its substitutions which omit a given letter must contain a transitive constituent of odd degree and hence of order $n!$ whenever $n > 4$. It may be noted in passing that the given considerations furnish also a proof of the known theorem that the continued product $(m + 1)(m + 2) \dots (2m)$ is always divisible by 2^m .

§ 4. *Group of the Square of a Determinant.*

In § 1 it was noted that the substitution group K on the n^2 elements of D composed of all the substitutions on these elements which transform D^2 into itself is of order $2(n!)^2$ and contains G invariantly. Its factors of composition are therefore 2, 2, 2, $n!/2$, $n!/2$. When $n = 2$, it is the octic group and

hence it is imprimitive. For all other values of n it is a simply transitive primitive group. In fact, it could not be doubly transitive, since the subgroup composed of all its substitutions which omit a given element has evidently two transitive constituents of degrees $2n - 2$ and $(n - 1)^2$ respectively. Moreover, it could not be imprimitive when n exceeds 3, since it contains the primitive group G which is also of degree n^2 . In the special case when $n = 3$, K is clearly again primitive since its subgroup composed of all its substitutions omitting one letter has two transitive constituents of degree 4, and 9 is not divisible by $4 + 1$.

One of the most interesting facts connected with the infinite system of simply transitive primitive groups represented by K is that these groups are of a very low class. It was noted near the close of § 2 that the class of a primitive group which is at least doubly transitive and does not include the alternating group cannot be less than its degree divided by 4; but when a primitive group is only simply transitive C. Jordan already observed that the ratio of the degree to the class has no upper limit. For the system of simply transitive primitive groups used by him as an illustrative example this ratio becomes however only about one half as large as in the present case since K is a simply transitive primitive group of degree n^2 and of class $2n$ whenever $n > 2$.*

As a transitive substitution group on the rows and columns of D , K is clearly the largest possible imprimitive group of degree $2n$ which has two systems of imprimitivity and hence it includes all the possible imprimitive groups of degree $2n$ which contain two such systems. As a transitive group on the $n!$ terms of D it is also imprimitive when $n > 2$, since it transforms the positive and negative terms of D into each other and involves a subgroup of index 2 which transforms these terms respectively among themselves. When $n > 3$ this subgroup is obtained by establishing a simple isomorphism between two simply transitive primitive groups, when $n = 3$ it is the direct product of two symmetric groups of degree n , and when $n = 2$ it is identity.

Since K contains a subgroup of index 2 which is simply isomorphic with the square of the symmetric group of degree n , it results directly that its smallest invariant subgroup is of

* Cf. W. A. Manning, *American Journal of Mathematics*, vol. 32 (1910), p. 256.

index 8 whenever $n > 4$, and that the corresponding quotient group is the octic group. Hence K involves exactly three subgroups of index 2 whenever n exceeds 4 and only two other invariants subgroups besides identity, viz., the mentioned subgroup of index 8 and one of index 4 corresponding to the invariant subgroup of order 2 of the octic group. These results apply also to the special case when $n = 3$.

TRANSLATION SURFACES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, April 27, 1918.)

1. If the rectangular coordinates of the points of a surface can be expressed in the parametric form

$$(1) \quad x_i = f_i(u) + g_i(v) \quad (i = 1, 2, \dots, n),$$

where f_i are functions of u alone and g_i functions of v alone, the surface is called a translation surface. It is seen that a translation can be found which will send any parameter curve $u = \text{const.}$ into any other one of the same system. The same is true of the curves $v = \text{const.}$ The surface (1) is also seen to be the locus of the mid-points of the lines joining the points of

$$(2) \quad C_1: x_i = 2g_i(u) \quad \text{to the points of} \quad C_2: x_i = 2f_i(v).$$

The character of the surface can then be determined, in a great measure, by the form and relative position of these two curves. Nearly all writers on surface theory* mention three facts concerning translation surfaces in 3-space:

(a) The generators of the developable which touches the surface along a curve $u = \text{const.}$ are tangent to the curves $v = \text{const.}$, or in other words the directions of the parameter curves passing through a given point are conjugate directions.

(b) There are surfaces which can be expressed in more than one way in the form (1).

* Darboux, *Théorie générale des Surfaces*, vol. 1, pp. 148, 340. Scheffers, *Theorie der Flächen*, vol. 2, pp. 188, 245.

(c) Minimum surfaces are always translation surfaces. The curves $u = \text{const.}$, $v = \text{const.}$, in this case, are minimum curves.

It is the object of the present note to examine translation surfaces in hyperspace as to the properties (a), (b), (c) and also to determine the relation between the curves C_1 and C_2 in order that (1) be a developable surface.

From (1) we see that the coordinates of the surface satisfy the linear partial differential equation of the second order

$$(3) \quad \frac{\partial^2 Z}{\partial u \partial v} = 0.$$

Segre* showed that if the coordinates of a surface satisfy a linear partial differential equation of the second order, then there are two directions through each point having the property (a) of conjugate directions. These are called the *characteristics* and are determined by the characteristic equation of the partial differential equation. The characteristic equation of (3) is

$$dudv = 0.$$

Therefore the parameter curves on (1) are characteristics. Hence *if (3) is the only partial differential equation of the second order which the coordinates of the surface satisfy, there are just two characteristics through each point and the surface cannot be expressed in more than one way in the form (1).* The form of equation (1) shows that the developables mentioned in (a) are cylinders.

2. *Ruled Translation Surfaces.*—Segre showed, in the paper referred to, that if the coordinates of a surface satisfy two linear partial differential equations of the second order the surface must either lie in a 3-space or else consist of the tangent lines to a twisted curve. I showed† that if the surface does not lie in a 3-space the two differential equations must be such that the pencil formed by them will contain only one equation of parabolic type. If now in addition to (3) the coordinates

* "Su una classe di superficie degl'iperspazii," ecc. *Atti di Torino*, 1907.

† C. L. E. Moore, "Surfaces in hyperspace which have a tangent line with three-point contact passing through each joint." This *BULLETIN*, vol. 18 (1912).

satisfy a second equation, it must be of the form

$$(4) \quad A \frac{\partial^2 Z}{\partial u^2} + B \frac{\partial Z}{\partial u} + C \frac{\partial Z}{\partial v} = 0.$$

Every equation of the pencil

$$A \frac{\partial^2 Z}{\partial u^2} + \lambda \frac{\partial^2 Z}{\partial u \partial v} + B \frac{\partial Z}{\partial u} + C \frac{\partial Z}{\partial v} = 0,$$

where λ is the parameter, will then be satisfied. The surface will then have an infinite number of characteristics. It is to be observed, however, that the characteristic equation is

$$A dv^2 - \lambda du dv = 0,$$

which for all values of λ has $dv = 0$ for one factor. This is the direction of the characteristic of (4), and I showed in the article referred to that this is the direction of three-point contact and that there is only one such direction passing through a point and hence this must be the direction of the rulings of the developable. We therefore conclude that however the surface is expressed in the form (1) the rulings must form one parameter system. The other parameter system can be any one-parameter family of curves traced on the developable. Equation (1) then takes the form

$$x_i = k_i u + g_i(v),$$

where k_i are constants. It is seen that the rulings are parallel and therefore the surface is a cylinder. Hence *cylinders are the only translation surfaces in hyperspace which can be expressed in the form (1) in more than one way.*

The coordinates of any ruled surface must satisfy a parabolic differential equation of the second order, and if it is a translation surface will satisfy two equations and therefore will be the same as above. Hence *cylinders are the only ruled surfaces of translation.*

3. *Developables.*—Let u and v be the arc length on the parameter curves. Then the element of arc on the surface becomes

$$ds^2 = du^2 + a_{12} du dv + dv^2,$$

where $a_{12} = \Sigma f_i' g_i'$. The formula for the Gaussian curvature then reduces to

$$(5) \quad \sqrt{a}G = \frac{\partial}{\partial v} \left[\frac{2}{\sqrt{a}} \frac{\partial a_{12}}{\partial u} \right],$$

where $a = 1 - a_{12}^2$. If $G = 0$, we have on integrating

$$(6) \quad a_{12} = \cos (U + V),$$

where U is an arbitrary function of u alone and V an arbitrary function of v alone. This is then the condition that the surface be developable, that is, have zero curvature. From the definition of a_{12} we see that (6) says that the angle between any tangent to C_1 and any tangent to C_2 is always expressible as a function of u plus a function of v . To determine the relation between C_1 and C_2 in order that (6) may be satisfied let γ_1 and γ_2 be the spherical representation of the curves C_1 and C_2 . (That is, γ_1 and γ_2 are the traces on the unit hypersphere, center at O say, made by lines through O parallel to the tangents.) The distance from a point of γ_1 to a point of γ_2 , measured on the sphere, will be equal to the angle between the corresponding tangents to C_1 and C_2 . Let u_i represent a series of points on γ_1 and v_i a series of points on γ_2 . Let θ_{ij} represent the distance from u_i to v_j . Then, if relation (6) is satisfied, we have

$$\theta_{ri} = U(u_r) - V(v_i), \quad \theta_{si} = U(u_s) - V(v_i).$$

Subtracting,

$$\theta_{ri} - \theta_{si} = U(u_r) - U(u_s).$$

Thus the difference of the distances from v_i to u_r and u_s is independent of v_i and hence v_i must lie in the locus of points the difference of whose distances from u_r and u_s is constant. This must be true for any values of r and s . Hence if γ_1 is an n -dimensional curve (does not lie in a space of lower dimensions), γ_2 must reduce to a point, which means that C_2 is a straight line. If however γ_1 lies in a linear space $S_k (k < n)$ passing through O , then (6) will be satisfied if γ_2 lies in the space S_{n-k} completely perpendicular to S_k . In this case the distance from any point of γ_1 to any point of γ_2 is $\pi/2$. C_1 and C_2 will then be any two curves lying in completely per-

pendicular spaces. In particular if γ_1 is a great circle on the hypersphere, then γ_2 could either lie in the completely perpendicular space or it could coincide with γ_1 , in which case (6) would be satisfied. The curves C_1 and C_2 for this last condition become any curves lying in parallel planes. The surface of translation would then reduce to a plane. If now γ_1 lies in a linear space S_k' which does not pass through O and γ_2 lies in the space S_{n-k}' passing through O and completely perpendicular to S_k' , then each point of γ_2 will be the same distance from all points of γ_1 but these distances will vary with the point of γ_2 . In this case then the distance is a function of v only and (6) will be satisfied. The curve C_2 will then be any curve lying in a space of $n - k$ dimensions and C_1 will be such that any tangent will make the same angles with any line in this space. If the space of C_2 be taken as a coordinate space the two curves must then have the form

$$(7) \quad \begin{aligned} C_1: \quad x_i &= g_i(u) & (i = 1, 2, \dots, k), \\ x_j &= a_j u & (j = k + 1, k + 2, \dots, n); \\ C_2: \quad x_i &= 0 & (i = 1, 2, \dots, k), \\ x_j &= f_j(v) & (j = k + 1, k + 2, \dots, n), \end{aligned}$$

where a_j are constants. We then have the results: (α) if C_1 and C_2 lie in parallel planes, the resulting translation surface is a plane; (β) if they lie in completely perpendicular planes, or (γ) if they have the form (7), the translation surface is a non-ruled developable. (δ) If one of the curves reduces to a straight line, the translation surface is ruled.

The surface generated by the midpoints of the lines joining the points of the two curves

$$\begin{aligned} x_1 &= 2a \cos u, & x_2 &= 2a \sin u, & x_3 &= x_4 = 0. \\ x_1 &= x_2 = 0, & x_3 &= 2b \cos v, & x_4 &= 2b \sin v \end{aligned}$$

is the rotation surface

$$x_1 = a \cos u, \quad x_2 = a \sin u, \quad x_3 = b \cos v, \quad x_4 = b \sin v$$

which is left invariant by all the rotations leaving the x_1x_2 and x_3x_4 planes invariant.*

* C. L. E. Moore, "Rotations in hyperspace," *Proceedings American Academy*, vol. 53 (1918).

4. *Covariant Derivatives*.—In discussing those properties of a surface that have to do with the normals or curvature of curves traced on the surface it is very convenient to make use of covariant derivatives instead of the ordinary second partial derivatives. If the surface is expressed in vector form, the covariant derivatives are always normal to the surface while the ordinary second partial derivatives are not. A second vector fundamental form, analogous to the second fundamental form in three dimensions can be written in terms of these covariant derivatives.* The vector equation of (1) is

$$\rho = \sum_1^n (f_i(u) + g_i(v))k_i,$$

where k_i are unit vectors parallel to the coordinate axes. Then

$$(8) \quad m = \frac{\partial \rho}{\partial u} = \sum f_i'(u)k_i, \quad n = \frac{\partial \rho}{\partial v} = \sum g_i'(v)k_i.$$

If u and v are arc lengths along the parameter curves,

$$(9) \quad m^2 = m \cdot m = \sum f_i'^2(u) = 1, \quad n^2 = n \cdot n = \sum g_i'^2(v) = 1.$$

The ordinary second partial derivatives are

$$(10) \quad p = \frac{\partial^2 \rho}{\partial u^2} = \sum f_i''(u)k_i, \quad q = \frac{\partial^2 \rho}{\partial u \partial v} = 0, \\ r = \frac{\partial^2 \rho}{\partial v^2} = \sum g_i''(v)k_i.$$

From these we obtain

$$(11) \quad m \cdot p = \sum f_i' f_i'' = 0, \quad m \cdot q = 0, \quad m \cdot r = \sum f_i' g_i'', \\ n \cdot p = \sum f_i'' g_i', \quad n \cdot q = 0, \quad n \cdot r = \sum g_i' g_i'' = 0.$$

In terms of these we can write the covariant derivatives†

$$(12) \quad y_{11} = (m \times n) \times (m \times n \times p) = \frac{1}{a} \begin{vmatrix} m & n & p \\ m^2 & m \cdot n & m \cdot p \\ n \cdot m & n^2 & n \cdot p \end{vmatrix},$$

* Wilson and Moore, "Differential geometry of two dimensional surfaces in hyperspace," *Proceedings American Academy*, vol. 52 (1916).

† Wilson and Moore, loc. cit., pp. 337-338.

where $a = 1 - (\Sigma f'g')^2$. The derivative y_{12} is obtained by replacing p by q in (12) and y_{22} is obtained by replacing p by r . It is seen that $y_{12} = 0$. This is a sufficient condition that the plane of the indicatrix (the locus of the end of the curvature vector of normal sections of the surface at a given point) pass through the surface point. Translation surfaces therefore are what Wilson and Moore called of the four-dimensional type.

5. If now the curves C_1 and C_2 lie in completely perpendicular spaces,

$$m \cdot n = \Sigma f'_i g'_i = 0.$$

This is the only surface of translation on which the parameter curves are orthogonal. The vector p will lie in the plane of C_1 , and r in the plane of C_2 , therefore

$$m \cdot r = \Sigma f'_i g''_i = 0, \quad n \cdot p = \Sigma f''_i g'_i = 0.$$

Hence we have for this type of developable

$$(13) \quad y_{11} = p, \quad y_{12} = 0, \quad y_{22} = r.$$

Thus p and r are normal to the surface and since they are the curvatures of the parameter curves we see that the curves $u = \text{const.}$, $v = \text{const.}$ are geodesics on the surface (curvature lies in the normal plane). Hence *on a non-ruled developable translation surface on which the generating curves are everywhere orthogonal those curves are geodesics.*

The area of the indicatrix is (Wilson and Moore, page 333)

$$2a^{3/2}\mu \times \delta = a_{12}y_{11} \times y_{22} + (a_{22}y_{11} - e_{11}y_{22}) \times y_{12};$$

and since $a_{12} = y_{12} = 0$, the area is zero. The indicatrix then reduces to a linear segment. Since y_{11} and y_{22} do not coincide in direction, this linear segment does not pass through the surface point. The curves $u = \text{const.}$, $v = \text{const.}$ are the Segre characteristics, and we know that when the indicatrix reduces to a linear segment the characteristics will be orthogonal and that the end of the curvature vector of these curves are the ends of the linear segment to which the indicatrix reduces. The indicatrix subtends a right angle at the surface point.

The vector mean curvature is half the sum of the vector

curvatures in two perpendicular directions passing through the point considered. This sum is independent of the pair of perpendicular directions chosen. Then in this case we can choose the parameter curves and we have

$$h = \frac{1}{2} \Sigma (f_i'' + g_i'') k_i,$$

from which we see that the locus of the end of the mean curvature vector is also a translation surface and in fact is a developable of the type of the original surface. The locus of the indicatrix as the point in question describes the whole surface is the locus of lines cutting the two curves

$$C_1'': x_i = 2f_i''(u), \quad C_2'': x_i = 2g_i''(v).$$

If the mean curvature is perpendicular to the indicatrix,

$$\Sigma f_i''^2 - \Sigma g_i''^2 = 0,$$

hence the curvature of C_1 and C_2 must be the same at all points and both constant, that is, they must be equal circles. If they have the same center the surface will be a rotation surface.

6. The second kind of non-ruled developable arises when

$$\begin{aligned} C_1: \quad x_i &= 2f_i(u) & (i = 1, 2, \dots, k), \\ x_j &= 2a_j u & (j = k+1, k+2, \dots, n); \\ C_2: \quad x_i &= 0 & (i = 1, 2, \dots, k), \\ x_j &= 2g_j(v) & (j = k+1, k+2, \dots, n), \end{aligned}$$

and the equation of the surface will be

$$\rho = \sum_1^k f_i(u) k_i + \sum_{k+1}^n (a_j u + g_j(v)) k_j.$$

In this case then we have

$$\begin{aligned} m \cdot p &= 0, \quad n \cdot p = 0, \quad m \cdot q = 0, \quad n \cdot q = 0, \quad n \cdot r = 0, \\ m \cdot r &= \Sigma a_j g_j''. \end{aligned}$$

If these values are substituted in (12), we have

$$y_{11} = p, \quad y_{12} = 0, \quad y_{22} = r - (m \cdot r)m + (m \cdot n)(m \cdot r)n.$$

Hence the curves $u = \text{const.}$ are geodesics on the surface, but in general the curves $v = \text{const.}$ are not. If however $m \cdot r = 0$ both sets of parameter curves are geodesics. Integrating this relation we have

$$\Sigma a_j g_j' = F = \text{const.},$$

and from what we saw previously concerning the relation § 3 this would require that the curve C_2 be a line and that the surface be ruled. Hence if C_1 and C_2 do not lie in completely perpendicular spaces and the translation surface is not ruled, both parameter systems cannot be geodesics on the surface.

The indicatrix for this type of surface does not reduce to a linear segment.

7. *Surfaces for Which C_1 and C_2 Coincide.*—If the curves C_1 and C_2 coincide, the surface (1) becomes the locus of the midpoints of the secants of a fixed curve. The surface is entirely similar to the same case in 3-space, except that in 3-space the fixed curve is an asymptotic line on the surface. Here the curve lies on the surface and is the locus of points at which the characteristics coincide. This curve has the property of an asymptotic line on a surface in 3-space.* The osculating plane of the curve is tangent to the surface and the tangent lines to this curve have three-point contact with the surface.† This is the only such line on the surface.

8. *Minimum surfaces.*—If we use the minimum curves on a surface as parameter lines, the coefficients in the first fundamental form are

$$a_{11} = m \cdot m = 0, \quad a_{22} = n \cdot n = 0, \quad a_{12} = m \cdot n.$$

The element of arc then becomes

$$ds^2 = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2 = 2a_{12}dudv.$$

The general formula for the mean curvature in terms of the covariant derivatives is

$$2h = \Sigma a^{(rs)} y_{rs},$$

where $a^{(rs)}$ is the complement of a_{rs} in the determinant $|a_{ij}|$

* Segre, loc. cit.

† C. L. Moore, "Surfaces in hyperspace, etc.," BULLETIN, vol. 18 (1912).

divided by a . In this case then

$$a^{(11)} = 0, \quad a^{(22)} = 0, \quad a^{(12)} = \frac{1}{a_{12}}$$

and the formula for h becomes

$$2h = \frac{1}{a_{12}} y_{12}.$$

The vanishing of the vector mean curvature is the necessary and sufficient condition for a minimum surface. Then, if the surface is minimum,

$$y_{12} = 0.$$

In 3-space if the minimum lines are taken as the parameter curves on a surface, the condition that the surface be minimum is the vanishing of the second coefficient in the second fundamental form. In hyperspace we have the same condition with respect to the vector second fundamental form. From (12) this condition becomes

$$y_{12} = (m \times n) \cdot (m \times n \times q) = 0.$$

The dot product is that used by Wilson and Lewis* and the vanishing here requires that q lie in the plane $m \times n$, that is

$$m \times n \times q = 0.$$

This is equivalent to saying that the coordinates must satisfy the differential equation

$$(14) \quad A \frac{\partial^2 Z}{\partial u \partial v} + B \frac{\partial Z}{\partial u} + C \frac{\partial Z}{\partial v} = 0,$$

which again amounts to saying that there is a linear relation connecting m , n , q

$$(15) \quad Aq + Bm + Cn = 0.$$

Differentiating the relations $m^2 = 0$, $n^2 = 0$, we see that

$$m \cdot q = 0, \quad n \cdot q = 0.$$

* "Space time manifold of relativity," *Proceedings Amer. Acad.*, vol. 48 (1912).

Multiplying (15) by m , we have

$$Cm \cdot n = Ca_{12} = 0.$$

Since $a_{12} \neq 0$, $C = 0$. Likewise multiplying by n we see that $B = 0$. Hence equation (14) becomes

$$A \frac{\partial^2 Z}{\partial u \partial v} = 0.$$

Hence, *the minimum surface is a surface of translation. The necessary and sufficient condition that a surface in hyperspace be a minimum surface is that the minimum lines on it are characteristics.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SOME ALGEBRAIC CURVES.

BY DR. JAMES H. WEAVER.

(Read before the American Mathematical Society, April 28, 1917.)

IN the following paper two algebraic curves are set up and some of their singularities are discussed. The author believes them to be new. At least a search through considerable of the literature on curves has failed to reveal them.

I.

Let there be any two distinct points A and B . Let the line joining A and B be drawn, and let the distance $AB = c$. Let there be drawn through A a line l_1 making an angle θ with AB , and let there be drawn through B a line l_2 making an angle $n\theta$ with AB (n an integer). We also consider that AB , l_1 , and l_2 are in one plane. Let the intersection of l_1 and l_2 be C . It is required to find the locus of C .

Let A be the origin and let AB be the x -axis. Then the equations of the lines l_1 and l_2 will be

$$(1) y = x \tan \theta, \quad (2) y = (x - c) \tan (n\theta)$$

respectively.

After eliminating θ from (1) and (2) we get

$$(3) \quad \begin{aligned} & x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \dots + (-1)^{n-1/2} x y^{n-1} \\ & = (x-c) \left[\binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} y^2 + \dots + (-1)^{n-1/2} y^{n-1} \right], \end{aligned}$$

where n is of the form $2k+1$. A similar form holds if n is of the form $2k$. The theory is the same in either case. (3) is then the equation representing the locus of C . Let us call this curve C_n . If in (2) we replace n by $n-r$, we obtain for (3) a curve of degree $n-r$, which we will call C_{n-r} . ($r = 1, 2, \dots, n-1$.)

From (3) it is evident that C_n has an $(n-1)$ -point at the origin. The equations of the $n-1$ tangents at this point will be given by

$$(4) \quad \binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} y^2 + \dots + (-1)^{(n-1/2)} y^{n-1} = 0.$$

The factors of (4) are

$$y - x \tan(k\pi/n) \quad (k = 1, \dots, n-1).$$

Therefore the tangents to C_n at the point A together with the x -axis divide the angular magnitude about A into $2n$ equal parts.

We will now consider the relation of the fixed point B to the curve C_n . Let us write (3) in homogeneous coordinates. It will then be

$$(5) \quad \begin{aligned} & x^n - \binom{n}{2} x^{n-2} y^2 + \dots + (-1)^{n-1/2} x y^{n-1} \\ & = (x - cz) \left[\binom{n}{1} x^{n-1} - \dots + (-1)^{n-1/2} y^{n-1} \right]. \end{aligned}$$

The first polar of this curve with respect to the point $B = (c, 0, 1)$ is

$$(6) \quad \begin{aligned} & x^{n-1} - \binom{n-1}{2} x^{n-3} y^2 - \dots + (-1)^{n-1/2} y^{n-1} \\ & = (x - cz) \left[\binom{n-1}{1} x^{n-2} - \dots + (-1)^{n-2/2} y^{n-2} \right]. \end{aligned}$$

This process may evidently be continued. We may then state the following

Theorem: The r th polar of B with respect to C_n is C_{n-r} .

II.

Again let there be three distinct points A , B , and C on the same straight line l , and through the point C let the line l_1 be drawn perpendicular to l . Let lines l_2 and l_3 be drawn through A and B respectively, and let l_2 and l_3 intersect on l_1 . Let l_2 make an angle α with l , and l_3 make an angle β with l , and let a line l_4 be drawn through B , making an angle $n\beta$ with l . Let l_2 and l_4 intersect in D . Then just as in section I, the equation representing the locus of D is

$$(7) \quad k \left[x^n - \binom{n}{2} x^{n-2} y^2 + \dots \right] \\ = (x - c) \left[\binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} y^2 + \dots \right],$$

where $k = (a - c)/a$ and $a = AC$, and $c = AB$.

It is then evident that the theorem in section I holds for the curve represented by equation(7).

OHIO STATE UNIVERSITY.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS

(Read before the American Mathematical Society, April 27, 1918.)

GIVEN the twisted cubic

$$(1) \quad x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0;$$

to show that the condition that it is a helix is precisely the condition that it is algebraically rectifiable.

If (1) is a helix, then T/R , the ratio of curvature to torsion, is constant. Denoting differentiation with respect to t by

primes, we have

$$\begin{aligned} x' &: a & 2bt & 3ct^2, \\ x'' &: 0 & 2b & 6ct, \\ x''' &: 0 & 0 & 6c, \end{aligned}$$

$$\begin{aligned} (x' | x') &= a^2 + 4b^2t^2 + 9c^2t^4, & (x'' | x'') &= 4(b^2 + 9c^2t^2), \\ (x' | x'') &= 2t(2b^2 + 9c^2t^2), & |x'x''x'''| &= 12abc, \\ (x'x'' | x'x'') &= (x' | x')(x'' | x'') - (x' | x'')^2 \\ &= 4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4). \end{aligned}$$

$$\frac{T}{R} = - \left(\frac{x'x'' | x'x''}{x' | x'} \right)^{3/2} \frac{1}{|x'x''x'''|}.$$

Since $|x'x''x'''|$ is constant, T/R is constant when and only when $(x'x'' | x'x'')/(x' | x')$ is constant. We thus have

$$4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4) \equiv \rho(a^2 + 4b^2t^2 + 9c^2t^4);$$

hence $\rho = 4b^2$ and $9a^2c^2 - 4b^4 = 0$. Conversely, for all values of a, b, c , $abc \neq 0$, for which

$$(2) \quad 9a^2c^2 - 4b^4 = 0,$$

T/R is constant—in particular, is equal to ∓ 1 , according as $2b^2 = \pm 3ac$ —and the cubic (1) is a helix.

If we had fixed our attention on another characteristic property of a helix, namely, that the tangent makes with a fixed direction a constant angle, we should have again derived the condition (2). The fixed direction—that of the axis of the cylinder on which the helix lies—is $(1/\sqrt{2}, 0, \pm 1/\sqrt{2})$ and the helix cuts the rulings of the cylinder under an angle of 45° .

That (2) is a necessary and sufficient condition that s , the arc of (1), is an algebraic function of t and hence that (1) is algebraically rectifiable follows from the fact that the integral

$$(3) \quad s = \int_{t_0}^t \sqrt{a^2 + 4b^2t^2 + 9c^2t^4} dt$$

is algebraic when and only when (2) holds. Hence the theorem: The twisted cubic (1) is algebraically rectifiable when and only when it is a helix.

SHORTER NOTICES.

A New Analysis of Plane Geometry Finite and Differential.

By A. W. H. THOMPSON. Cambridge, University Press, 1914. 16 + 120 pages. Price 7 shillings.

THIS book is an exposition of a system of notation for plane metric geometry. Points are represented by Latin letters, lines by Greek letters; $\overline{ab}$ denotes the line determined by the points a and b , and $\alpha\beta$ denotes the point determined by the lines α and β ; (ab) , $(\alpha\beta)$, and $(a\beta)$ denote respectively the distance between a and b , the angle between α and β and the perpendicular distance between a and β . The author's problem is now one of reductional computation—to express in terms of these measures of two elements any measure of elements derived from points and lines by intersections and joins, vectorial constructions, and equational relations. This reduction is carried out not only for such measures as areas of triangles and the trigonometric functions but for differentiation and integration. The author claims that his method is superior to that of coordinate geometry in the matter of sign. There are nearly two hundred examples in the text, the majority of them being grouped at the end.

G. H. GRAVES.

An Elementary Course in Differential Equations. By EDWARD J. MAURUS. Ginn and Company, 1917. vi + 51 pp.

THIS is a collection of about 350 problems covering ordinary differential equations up to solution in series. Very brief explanations of the methods of formation and of solving differential equations are given, and the applications are only hinted at.

C. F. CRAIG.

An Introduction to Statistical Methods. By HORACE SECRIST. New York, Macmillan, 1917. xxi + 482 pp. \$2.00

As stated on the title page, this is "A text book for college students, a manual for statisticians and business executives." The book is descriptive rather than mathematical in character, making its appeal to the general reader through its discussion of methods and purposes.

Sufficient illustrations are given to make the meaning clear. The discussion is detailed, almost wordy at times. This very wordiness makes the book of especial value to the beginner. Each chapter ends with a summary and a list of references to other standard works on the same and allied subjects.

The chapter on averages discusses at some length the arithmetic mean, the mode, and the median and omits the geometric mean, and the mean given by $\sqrt{\Sigma x^2/n}$, and combinations of these averages.

Illustrative matter is mainly from the economic field. A few illustrations are from agriculture. References are made, however, to other fields, such as biology, psychology, genetics.

Although non-mathematical, the book is of interest to a mathematician from the point of view of the applications. This is a good book for a beginner and at the same time useful to one already initiated into the study of statistics.

W. V. LOVITT.

Cours de Mécanique. Vol. III. By LÉON LECORNU. Paris, Gauthier-Villars, 1918. 669 pp.

THIS is the third and last volume of a treatise on mechanics for use in l'Ecole Polytechnique. The first was reviewed in the BULLETIN for April, 1915, and the second, November, 1917.

The present volume is devoted to applications of mechanics to engineering and consists of five parts (parts X to XIV of the complete course). The subjects considered are strength of materials, hydraulics, thermodynamics, theory of machines, and a brief discussion of the problems involved in aviation.

It is interesting to compare the preface to the third volume with the preface to the first, which appeared in 1914. Just before the war, Professor Lecornu, in speaking of the course in mechanics as a whole, referred to the necessity for resisting a demand for the teaching of practical applications and expressed his firm belief that the course in l'Ecole Polytechnique should be purely theoretical. In support of this position he quoted General Langlois. "The officers who leave the school at the end of one year are, in general, inferior to their comrades in the matter of studying logically and deeply a scientific question of tactics or organization. The method of work indispensable to every man of action demands imperiously the study of a science to its foundations, a study which makes

the intellect supple and develops a habit of logical deduction necessary to one who commands."

In the preface to the third volume, written after three and a half years of war, the author explains that the experiences through which the country is passing have influenced him to extend the treatment of applied mechanics beyond the course as it is now given. He foresees, after peace is reestablished, a profound transformation in scientific study, which will adapt it more directly to the realities of life (*réalités de la vie*). He refers to the action of L'Académie des Sciences in deciding (January, 1918) to admit a certain number of representatives of industry and predicts that l'Ecole Polytechnique will reduce the time now given to abstract theory and increase the time allotted to applied work.

W. R. LONGLEY.

Electric and Magnetic Measurements. By CHARLES MARQUIS SMITH. (Edited by E. R. Hedrick.) New York, Macmillan, 1917. xii + 373 pp.

FROM the point of view of the mathematical reader the interest of this book is purely incidental. Moreover, it is incidental to study not in any general field but only in the special field of the mathematical theory of electricity and magnetism. In fact, the book belongs to a series of texts on topics in engineering. As such a detailed review of it is out of place in this BULLETIN. It is well, however, at this time when more interest is being manifested in applied mathematics than heretofore in this country to have attention directed to a convenient description of the instruments and methods by means of which are measured the quantities involved in the theory of electricity and magnetism, a subject which has lent itself in a remarkable way to precise mathematical treatment. The book under consideration furnishes in convenient form what such a mathematical reader will desire. The fact that it was written for engineering students does not interfere with this use of the book.

R. D. CARMICHAEL.

NOTES.

ON account of war conditions the Southwestern Section of the American Mathematical Society will not hold its meeting this year. No eastern meeting will be held until April 26, 1919. The Chicago meeting in the Christmas holidays will be the annual meeting of the Society for the election of officers and other members of the Council, and will be especially marked by the retiring address of President L. E. DICKSON.

THE third summer meeting of the Mathematical Association of America was held at Dartmouth College on Thursday, Friday, and Saturday, September 5-7, 1918, immediately following the summer meeting of the American Mathematical Society at the same place. A joint dinner of the two organizations, on Thursday evening, was attended by fifty-six members and friends. At the joint session on Friday morning Professor A. G. WEBSTER gave an address on "Mathematics of warfare." The retiring address of President CAJORI: "Plans for a history of mathematics of the nineteenth century," was read by Dean G. D. Olds. The other papers on the regular programme were: "The teaching of curve tracing," by F. W. OWENS; "A formula in combinatorial analysis," by J. W. YOUNG; "Trigonometric functions—of what?," by W. B. CARVER; "Firing data at Yale," by J. K. WHITTEMORE; "A combined course in mathematics for college freshmen," by A. S. GALE; Report of the committee on mathematical requirements, by J. W. YOUNG, chairman; "Some experiments in the teaching of descriptive geometry," by F. L. KENNEDY. A very great interest was shown in the new courses for military training in the colleges, and nearly all the members present cooperated actively with a committee appointed to prepare a draft of such courses for submission to the authorities. The total attendance at the sessions numbered seventy-one, including forty-two members. Fourteen new members were elected.

THE July number of the *Transactions of the American Mathematical Society* contains the following papers: "Singular points

of analytic transformations," by W. F. OSGOOD; "Space involutions defined by a web of quadrics," by VIRGIL SNYDER and F. R. SHARPE; "On the location of the roots of the Jacobian of two binary forms, and of the derivative of a rational function," by J. L. WALSH; "Sets of independent generators of a substitution group," by G. A. MILLER; "The problem of Mayer with variable end points," G. A. BLISS.

PROFESSORS O. D. KELLOGG, of the University of Missouri, and MAX MASON, of the University of Wisconsin, are engaged in government service at the Naval Experimental Station at New London, Conn.

THE following university teachers are giving instruction in mathematics at the field artillery central officers' training school at Camp Zachary Taylor, Ky.: Dr. L. R. FORD, Harvard University; Mr. R. W. BARNARD and Professor T. H. HILDEBRANDT, University of Michigan; Mr. J. D. ESHLEMAN, University of Rochester; Dr. T. R. HOLLCROFT, Columbia University; Dr. C. E. WILDER and Professor C. H. YEATON, Northwestern University; Mr. H. E. WOLFE, University of Indiana. Mr. R. E. MOORE, University of Wisconsin, died October 2.

PROFESSOR G. N. WATSON, of Trinity College, Cambridge, and University College, London, has been appointed to the professorship of mathematics in the University of Birmingham, in succession to Professor R. S. Heath.

At the University of Saskatchewan assistant professor L. L. DINES has been promoted to a full professorship of mathematics.

DR. R. M. WINGER, of the University of Oregon, has been appointed assistant professor of mathematics in the University of Washington.

DR. F. W. BEAL, of the University of Pennsylvania, has been appointed professor of mathematics in the University of Tennessee.

DR. L. E. WEAR, of the University of Washington, has been appointed associate professor of mathematics in Throop College.

DR. J. R. KLINE, of the University of Pennsylvania, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

PROFESSOR PAOLO PIZZETTI, of the University of Pisa, died April 14, 1916, at the age of fifty-eight years.

BOOK CATALOGUES: Galloway and Porter, Cambridge, England, catalogue 93, about 100 titles in mathematics and physics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIANCHI (L.). Lezioni sulla teoria dei gruppi continui finiti di trasformazioni. Pisa, E. Spoerri (F. Mariotti), 1918. 8vo. 6 + 590 pp. L. 25.00

CANDIDO (G.). La risultante di due quadratiche. Livorno, tip. R. Giusti, 1918. 4to. 26 pp.

CHISINI (O.). See ENRIQUES (F.).

COLPITTS (J. T.). See ROBERTS (M. M.).

ENRIQUES (F.). Conferenze sulla geometria non-euclidea, per cura del O. Fernandez. Bologna, Zanichelli, 1918. 8vo. 46 pp. L. 3.00

———. Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, pubblicate per cura O. Chisini. Volume 2. Bologna, Zanichelli, 1918. 8vo. 713 pp. L. 25.00

FERNANDEZ (O.). See ENRIQUES (F.).

GAUSS (C. F.). Werke. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. 10ter Band, 1te Abteilung. Leipzig, Teubner, 1917. 4to. 586 pp.

HUYGENS (C.). Du calcul dans les jeux de hasard, rédigé par D. J. Korteweg. La Haye, Martinus Nijhoff, 1917. 4to. 179 pp.

KORTEWEG (D. J.). See HUYGENS (C.).

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte 1: Calcolo differenziale. 4a edizione riveduta. (Manuale Hoepli.) Milano, Hoepli, 1918. 24mo. 12 + 313 pp. L. 4.50

PEZZO (P. DEL). Principi di geometria proiettiva: lezioni dettate nell' università di Napoli nell' anno 1917-1918. Napoli, tip. B. De Rubertis, 1918. 8vo. 116 pp.

ROBERTS (M. M.) and COLPITTS (J. T.). Analytic geometry. New York, Wiley, 1918. 10 + 229 pp. \$1.60

TORELLI (G.). Lezioni di calcolo infinitesimale date nella r. università di Napoli. Napoli, tip. Accademia r. delle Scienze fisiche e matematiche, 1918. 8vo. 4 + 424 pp. L. 20.00

II. ELEMENTARY MATHEMATICS.

BOSTON, DEPARTMENT OF EDUCATIONAL INVESTIGATION AND MEASUREMENT. Arithmetic; the value to the teacher, to the principal and to the superintendent of individual and class record from standard tests. (Bulletin 13; School documents 22, 1917.) Boston, School Committee, 1917. 8vo. 83 pp.

CLARK (J. R.). See RUGG (H. O.).

CORRELL (F. E.) and FRANCIS (M. E.). Exercises in arithmetic. Waverly, Iowa, Francis-Correll Company, 1917. 8vo. 71 pp. \$0.25

DAVIS (R.). Business practice in elementary schools. (Harvard Studies in Education.) Cambridge, Harvard University Press, 1917. 8vo. 31 pp. Paper. \$0.30

FRANCIS (M. E.). See CORRELL (F. E.).

IANNELLI (G.). Di un metodo pratico per la trisezione dell' angolo. Palermo, tip. Pontificia, 1918. 4to. 9 pp. + 1 tavola.

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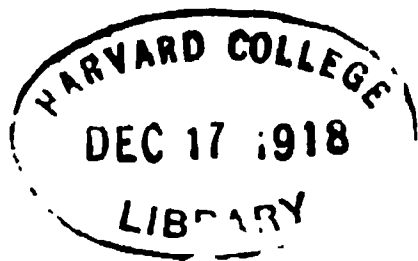
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GENERAL ASPECTS OF THE THEORY OF
SUMMABLE SERIES.

BY PROFESSOR R. D. CARMICHAEL.

(Read at the Chicago Symposium of the American Mathematical Society,
April 12, 1918.)§1. *General Considerations Relating to the Sum of an
Infinite Series.*

IN 1811 Fourier* read before the Paris Academy a memoir which contained an acceptable definition of convergence of an infinite series; but this work remained unpublished for eight or nine years. In 1817 Bolzano stated a precise definition of convergence. Independently in 1821 Cauchy also formulated the definition in an exact manner. He and Abel insisted so forcefully upon the necessity of the distinction between convergence and divergence and the danger in employing divergent series that the latter came into such disrepute as not to be studied systematically for nearly three quarters of a century. For a long time no one saw how to obviate the difficulties pointed out so incisively by those who first recognized the pitfalls in the use of divergent series. And yet both Abel and Cauchy, the leading instigators, had misgivings† as to the justice of the decision by which these series were banished from the mathematical community and they were given up as friends who had done some things well but could not be trusted because they had also done some things ill.‡

Certain difficulties, however, still remain when one tries to treat convergent series independently of any reference to divergent series, as we shall show more fully in a moment.

In the first attempt to formulate a suitable definition of

* For references relating to the first paragraph see *Encyclopédie des Sciences mathématiques*, I, 1, pp. 211-214.

† See quotations in Bromwich's *Infinite Series*, 1908, p. 264. Indeed Cauchy himself showed how the celebrated series of Stirling in the theory of the gamma function could be used in a legitimate way for purposes of numerical computation.

‡ An interesting and valuable discussion of several topics in the theory of divergent series and continued fractions will be found in Van Vleck's lectures at the Boston Colloquium in 1903, published in 1905. In these lectures some topics are treated to which we do not refer in this paper.

the sum of the infinite series

$$(1) \quad u_0 + u_1 + u_2 + \dots$$

it is natural to employ with Cauchy the sum s_n of the first $n + 1$ terms, namely,

$$(2) \quad s_n = u_0 + u_1 + \dots + u_n,$$

and to say that the series (1) has the sum s in case $\lim_{n \rightarrow \infty} s_n$ exists and has the finite value s . But there is no good reason why we should confine attention to this definition alone when our researches so often bring us face to face with series not possessing a sum in this sense.

As a matter of fact one does not have to go far to find the inadequacy of this definition. One of the leading tasks in developing the theory of infinite series is to determine the fundamental laws of operation according to which one may compute with them. Certain of these are at once obvious, as for instance those associated with the introduction or removal of a finite number of terms, the term by term addition of two series, and the multiplication of a series term by term by a constant. But if one undertakes to form the product of two series the case is different. Consider the product of (1) by the series

$$v_0 + v_1 + v_2 + \dots$$

Whenever the two series are absolutely convergent and have the sums u and v respectively, it may be shown without difficulty that the Cauchy product series*

$$(3) \quad w_0 + w_1 + w_2 + \dots,$$

where

$$w_n = u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0,$$

converges and has a sum w which is equal to uv . (A like conclusion is true also under certain less restrictive hypotheses.) But the mere convergence of the u -series and the v -series does not necessitate the convergence of the w -series.

* This definition of product of two series is that most naturally associated with power series; but there is nothing inherently essential in it. The product of two series may in fact be defined in any one of a variety of ways, with consequent variations in the theory. On account of the importance of power series, however, it is desirable to have a theory of infinite series adequate for the case in which multiplication is defined in accordance with the Cauchy product formula.

Starting from the problem of forming the product of two convergent series, Cesàro* in 1890 was led to an investigation of what he called indeterminate series. He showed that, when the u -series and v -series above converge to the sums u and v respectively, then the corresponding w -series has the property expressed in the relation

$$(4) \quad \lim_{n \rightarrow \infty} \frac{W_0 + W_1 + \cdots + W_n}{n+1} = uv,$$

where

$$W_n = w_0 + w_1 + \cdots + w_n.$$

Thus he was led to say that series (3) has the *sum* w whenever the limit in (4) exists and has the finite value w . (It is easy to show—see §2 below—that this new definition assigns to every convergent series the same sum as the usual definition.)

By the introduction of this definition the multiplication problem in which Cesàro was interested became enlarged. Suppose now that the series u and v have sums in the new sense; what can be said of the product series w ? Following up this question, Cesàro was led to extend further the definition of sum of an infinite series. Thus when we have for a finite s the relation

$$(5) \quad s = \lim_{n \rightarrow \infty} \frac{S_n^{(r)}}{D_n^{(r)}},$$

where

$$\begin{aligned} S_n^{(r)} &= s_n + r s_{n-1} + \frac{r(r+1)}{2!} s_{n-2} \\ &\quad + \cdots + \frac{r(r+1) \cdots (r+n-1)}{n!} s_0 \\ &= u_n + (r+1)u_{n-1} + \frac{(r+1)(r+2)}{2!} u_{n-2} \\ &\quad + \cdots + \frac{(r+1)(r+2) \cdots (r+n)}{n!} u_0, \\ D_n^{(r)} &= \frac{(r+1)(r+2) \cdots (r+n)}{n!}, \end{aligned}$$

it is said that the series (1) has the sum s . We then say that the series (1) is *summable* (Cr) to the sum s . The least value of r (assumed to be an integer) for which the limit in (5)

* *Bulletin des Sciences mathématiques*, ser. 2, vol. 14 (1890), pp. 114–120.

† *Math. Annalen*, vol. 20 (1882), pp. 535-549. This limit also appeared in the work of Euler and Abel. More recently it has been prominent in the literature of summable series.

where the a_{ij} are (real or complex) constants. In fact, the functions t_n of n in (6) and (7) are identical in case

$$\begin{aligned}
 a_{nn} &= c_{nn}, \\
 a_{n,n-1} &= c_{nn} + c_{n,n-1}, \\
 a_{n,n-2} &= c_{nn} + c_{n,n-1} + c_{n,n-2}, \\
 &\dots \dots \dots \\
 a_{n0} &= c_{nn} + c_{n,n-1} + \dots + c_{n0}.
 \end{aligned}
 \tag{8}$$

These relations are obviously equivalent to the following:

$$\begin{aligned}
 c_{nn} &= a_{nn}, \\
 c_{n,n-1} &= a_{n,n-1} - a_{n,n}, \\
 c_{n,n-2} &= a_{n,n-2} - a_{n,n-1}, \\
 &\dots \dots \dots \\
 c_{n0} &= a_{n0} - a_{n1}.
 \end{aligned}
 \tag{8'}$$

Certain other particular cases of summation by the method of mean values should be mentioned.

In 1907 Knopp* generalized Cesàro's definition of sum by taking for the sum s of (1) the value

$$s = \lim_{n \rightarrow \infty} \frac{\Gamma(r+1)}{n^r} S_n^{(r)},$$

where

$$S_n^{(r)} = \sum_{k=0}^n \frac{\Gamma(r+n-k+1)}{\Gamma(r+1)\Gamma(n-k+1)} u_k,$$

the symbol Γ denoting the gamma function. For positive integral values of r this is the same as Cesàro's definition.

De la Vallée Poussin† in 1908 defined the sum of (1) to be the number s in case

$$(9) \quad \lim_{n \rightarrow \infty} \left(u_0 + \sum_{k=1}^n \frac{n(n-1) \cdots (n-k+1)}{(n+1)(n+2) \cdots (n+k)} u_k \right)$$

exists and has the finite value s .

In 1909 Ford‡ gave a generalization of Cesàro's method of

* *Sitzungsber. d. Berliner Math. Gesellschaft*, Nov., 1907, pp. 1-12.

† *Belg. Bulletin d. Sciences*, 1908, pp. 193-254. See also a generalization by Kogbetliantz, *Paris C. R.*, vol. 164 (1917), pp. 510-513, 626-628, 778-780.

‡ This BULLETIN, vol. 15 (1909), pp. 439-444.

summation in accordance with which the sum of (1) is said to have the value s ,

$$(10) \quad s = \lim_{n \rightarrow \infty} \frac{f_p(n, r)s_0 + f_p(n-1, r)s_1 + \cdots + f_p(0, r)s_n}{f_p(n, r) + f_p(n-1, r) + \cdots + f_p(1, r) + f_p(0, r)},$$

provided this limit exists, the function $f_p(n, r)$ being defined by the relation

$$(11) \quad f_p(n, r) = \frac{r(r+1)}{r + \log_p n} \prod_{k=2}^n \frac{r + \log_p k}{\log_p k}$$

where $\log_0 n = n$ and $\log_p n = \log(\log_{p-1} n)$, $p = 1, 2, 3, \dots$. For $p = 0$ this reduces to Cesàro's definition.

Chapman* in 1911 developed the theory of Cesàro's mean value process when the order r of summability is any real number other than a negative integer, employing for this purpose the Cesàro formulæ which we have given above (see equation (5) and those immediately following it) but without Cesàro's restriction that r shall be integral.†

Silverman (l. c., page 37) says that (1) is φ -summable to the sum s in case

$$\lim_{n \rightarrow \infty} \frac{\varphi_0 s_0 + \varphi_1 s_1 + \cdots + \varphi_n s_n}{n+1} = s, \quad \lim_{n \rightarrow \infty} \varphi_n = 1.$$

On account of needs arising in the study of Dirichlet series M. Riesz‡ was led to define the sum of (1) as the finite number s in case

$$(12) \quad \lim_{\omega \rightarrow \infty} \omega^{-r} \sum_{\lambda_k < \omega} (\omega - \lambda_k)^r u_k = s,$$

where $\lambda_0 \geq 0$, $\lambda_1, \lambda_2, \dots$ is a sequence of distinct real numbers tending monotonically to infinity with n and the sum for each ω is taken for all $\lambda_k < \omega$. When the limit in (12) exists and is finite we shall say (with Hardy§) that (1) is summable $(R\lambda r)$, that is, summable by Riesz's means of type λ and order r .

It will be observed that this definition differs from the pre-

* *Proc. London Math. Society*, ser. 2, vol. 9 (1911), pp. 369-409.

† An extension of Cesàro's definition to double series has been given by C. N. Moore, *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 73-104.

‡ *Paris C. R.*, vol. 152 (1911), pp. 1651-1654. See also an earlier note by Riesz in *Paris C. R.*, vol. 149 (1909), 18-20. This paper contains an error which is corrected in the later communication.

§ *Proc. London Math. Society*, ser. 2, vol. 8 (1909), pp. 301-320.

ceding ones in that the limit is taken with respect to the continuous variable ω rather than with reference to a variable n running over the set of non-negative integers. Following out the suggestion thus arising, one might modify the definition associated with (6) and (7) so that the limit with respect to n shall be replaced by a limit with respect to a continuous variable. Compare the discussion associated with (13).

Several considerations have led investigators to introduce other definitions of sum than those associated with the method of mean values with finite reference. In his now classic researches Borel was influenced by the fact that the method of Cesàro suffices to sum only a relatively restricted class of series (see §5 below). Accordingly he introduced* *the method of mean values with infinite reference*. In a form convenient for our purposes the method may be stated thus:† *The series (1) is said to have the sum t in case $\lim_{n \rightarrow \infty} t_n$ exists and has the finite value t , where*

$$(13) \quad t_n = \sum_{k=0}^{\infty} c_{nk} s_k,$$

the c_{ij} being (real or complex) constants.

For an alternative definition one may define t_n by the relation

$$t_n = \sum_{k=0}^{\infty} a_{nk} u_k$$

where the a_{ij} are (real or complex) constants.

In order to specialize the definition associated with (13) into a conveniently workable form one may take for c_{nk} the value $c_k n^k / \varphi(n)$, where

$$\varphi(a) = c_0 + c_1 a + c_2 a^2 + c_3 a^3 + \dots,$$

and modify the limit operation by replacing n by the continuous variable a so as to have for the sum of (1) the (finite) value t , where

$$t = \lim_{a \rightarrow \infty} \frac{c_0 s_0 + c_1 a s_1 + c_2 a^2 s_2 + \dots}{\varphi(a)}.$$

* *Leçons sur les Séries divergentes*, 1901, pp. 91–98. Here will be found references to Borel's earlier work on this subject.

† Here the variable n may run over the range of integers or over a continuous range.

For detailed study Borel chooses* for $\varphi(a)$ the function e^a , so that he defines the sum t of (1) by the more special relation

$$(14) \quad t = \lim_{a=\infty} e^{-a} \left(s_0 + s_1 \frac{a}{1} + s_2 \frac{a^2}{2!} + s_3 \frac{a^3}{3!} + \cdots \right).$$

This definition of sum gives rise to what Borel calls the *exponential method* of summation of series. It is obviously a special case of the method of mean values with infinite reference.

For the sum t of (1) LeRoy† has taken the value

$$(15) \quad t = \lim_{x \rightarrow 1-0} \sum_{k=0}^{\infty} \frac{\Gamma(kx+1)}{\Gamma(k+1)} u_k,$$

provided that this limit exists and is finite. He was led to this definition through a consideration of the problem of analytic continuation of a function defined by a power series.

It is obvious that one may extend the definition in (13) by considering the coefficients c_{ij} or a_{ij} to be functions of parameters $x_1, x_2, \dots, x_r$ and taking the sum of (1) to be

$$\lim_{x_1=l_1} \lim_{x_2=l_2} \cdots \lim_{x_r=l_r} \lim_{n=\infty} t_n$$

when this repeated limit exists and is finite. For a treatment of some such definitions see the papers referred to at the end of this section, especially those of Hardy and Chapman, Chapman, and Smail.

An easy and natural step‡ leads one from Borel's exponential method of summation to his integral method. Denoting by $s(a)$ the function

$$s(a) = s_0 + s_1 \frac{a}{1} + s_2 \frac{a^2}{2!} + s_3 \frac{a^3}{3!} + \cdots,$$

one has from (14) the relation

$$t - u_0 = [e^{-a}s(a)]_0^{\infty} = \int_0^{\infty} \frac{d}{da} [e^{-a}s(a)] da,$$

* Interesting applications are also made of certain other cases. See the papers referred to in the second preceding footnote. In particular, considerable treatment (*Séries divergentes*, pp. 129 ff.) is given of the case in which $\varphi(a) = e^a$.

† *Annales Fac. Sci. Toulouse*, ser. 2, vol. 2 (1900), pp. 317-430; see especially pp. 327-328.

‡ Borel, l. c., pp. 97-100. On Borel's generalizations of the notion of limit see also Hanni, *Monatshefte für Mathematik und Physik*, vol. 12 (1901), pp. 265-289.

provided that

$$\lim_{a \rightarrow 0} e^{-a}s(a) = u_0.$$

But

$$\begin{aligned} \frac{d}{da}[e^{-a}s(a)] &= e^{-a}[s'(a) - s(a)] \\ &= (s_1 - s_0) + \frac{(s_2 - s_1)a}{1!} + \frac{(s_3 - s_2)a^2}{2!} + \dots \\ &= u_1 + u_2 \frac{a}{1!} + u_3 \frac{a^2}{2!} + \dots \end{aligned}$$

Denoting by $\bar{u}(a)$ the function in the last member and integrating by parts, we have

$$\begin{aligned} t - u_0 &= \int_0^\infty e^{-a}\bar{u}(a)da \\ &= \left[e^{-a} \int_0^\infty \bar{u}(a)da \right]_0^\infty + \int_0^\infty e^{-a} \left[\int_0^\infty \bar{u}(a)da \right] da; \end{aligned}$$

whence it follows that

$$(16) \quad t = \int_0^\infty e^{-a}u(a)da,$$

where

$$u(a) = u_0 + \frac{u_1 a}{1!} + \frac{u_2 a^2}{2!} + \frac{u_3 a^3}{3!} + \dots$$

In the detailed development of the theory it is assumed by Borel that the series (1) is such that the associated function $u(a)$ is an entire function. Then when the integral in (16) exists he takes its value t to be the sum of (1) and says that the series is summable to the sum t .

Furthermore, if the integrals

$$\int_0^\infty e^{-a}|u(a)|da, \quad \int_0^\infty e^{-a}|u^{(\lambda)}(a)|da \quad (\lambda = 1, 2, 3, \dots),$$

exist, where λ is an index of differentiation, then Borel says that (1) is absolutely summable.

Borel has also employed a generalization of his definition (16) in which

$$t = \int_0^\infty e^{-a}u_p(a)da,$$

where

$$\begin{aligned} u_p(a) = & (u_0 + u_1 + \cdots + u_{p-1}) \\ & + (u_p + u_{p+1} + \cdots + u_{2p-1}) \frac{a}{1!} \\ & + (u_{2p} + u_{2p+1} + \cdots + u_{3p-1}) \frac{a^2}{2!} + \cdots \end{aligned}$$

For a discussion of the definitions of LeRoy see §7 below.

It is convenient to insert here also references to the methods of Euler,* Buhl,† Stieltjes,‡ Hardy and Chapman,§ Chapman,|| Barnes,¶ Smail,** James,†† C. N. Moore,‡‡ Servant,§§ W. H. Young,||| and Cunningham.¶¶

For the sake of unity and economy of space we shall not treat the question of uniform summability, particularly since the notion of uniformity enters in essentially an obvious way.

§2. Regularity of a Definition of Sum by the Method of Mean Values with Finite Reference.

A definition of sum of an infinite series is said to be *regular**** if it assigns to every convergent series the same sum as the usual definition. It is desirable (and natural) to confine attention to those definitions which are regular in this sense.

We shall now determine necessary and sufficient conditions that the general definition of sum by the method of mean

* Euler's treatment of divergent series (Inst. Calc. Diff., Pars II, Cap. I) depends on a transformation resulting in convergent series when applied to certain classes of divergent series. See the treatment of this method by Bromwich, *Infinite Series*, pp. 302-310.

† *Bulletin des Sciences mathématiques*, vol. 42 (1907), pp. 340-346; *Journal de Mathématiques*, ser. 6, vol. 4 (1908), pp. 367-377.

‡ *Annales de Toulouse*, 8J, pp. 1-122; 9A, pp. 1-47; 1894-1895.

§ *Quarterly Journal of Mathematics*, vol. 42 (1911), pp. 181-215. A general class of definitions forms the subject matter of this paper. A brief statement of the guiding principle is given in §8 below.

|| *Quarterly Journal of Mathematics*, vol. 43 (1911), pp. 1-52.

¶ *Phil. Transactions Royal Society*, 199A (1902), pp. 411-500.

** Columbia dissertation, 1913.

†† Columbia dissertation, 1917.

‡‡ *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 299-330; vol. 14 (1913), pp. 73-104.

§§ *Annales de Toulouse*, ser. 2, vol. 1 (1899), pp. 117-175.

||| *Leipziger Berichte*, vol. 63 (1911), pp. 369-387.

¶¶ *Proc. London Math. Society*, ser. 2, vol. 3 (1905), pp. 157-169.

*** The term "regular" is used in essentially the sense of the text by Hurwitz and Silverman, *Transactions Amer. Math. Society*, vol. 18 (1917), pp. 1-20.

values with finite reference shall be regular. Employing the notation of equation (6) and using the definition of regularity, we see that we must have $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n$ for every convergent sequence $s_0, s_1, s_2, \dots$. Then if we take $s_k = 1$ and $s_n = 0$ when $n \neq k$, we have $\lim_{n \rightarrow \infty} c_{nk} = 0$ for every k . Again, taking $s_0 = 1 = s_1 = s_2 = \dots$, we find the second of the following two necessary conditions:

$$(17) \quad \lim_{n \rightarrow \infty} c_{nk} = 0, \quad \lim_{n \rightarrow \infty} (c_{n0} + c_{n1} + \dots + c_{nn}) = 1.$$

In what follows we employ these conditions.

Consider the superior limit

$$\limsup_{n \rightarrow \infty} A_n, \text{ where } A_n = |c_{n0}| + |c_{n1}| + \dots + |c_{nn}|.$$

In view of (17) its value must be either infinity or a positive number not less than unity. Suppose first that its value is infinity. We shall determine a sequence $s_0, s_1, s_2, \dots$ approaching zero such that t_n does not approach a finite limit as n approaches infinity. Let α be a number greater than 1 and choose n_1 so that $A_{n_1} > \alpha^2$. Then put*

$$s_i = \frac{1}{\alpha} \frac{|c_{n_1 i}|}{c_{n_1 i}} \quad (i=0, 1, \dots, n_1).$$

Then $|t_{n_1}| > \alpha$. Choose n_2 , greater than n_1 , so that

$$|c_{n_2 0}| + \dots + |c_{n_2 n_1}| < \alpha, \quad A_{n_2} > \alpha^4 + 2\alpha^2.$$

Then put

$$s_i = \frac{1}{\alpha^2} \frac{|c_{n_2 i}|}{c_{n_2 i}} \quad (i = n_1 + 1, \dots, n_2).$$

It is easy to see that $|t_{n_2}| > \alpha^2$; for

$$\begin{aligned} |t_{n_2}| &\geq \left| \sum_{i=n_1+1}^{n_2} c_{n_2 i} s_i \right| - \sum_{i=0}^{n_1} |c_{n_2 i} s_i| \\ &\geq \left| \frac{1}{\alpha^2} A_{n_2} - \frac{1}{\alpha^2} \sum_{i=0}^{n_1} |c_{n_2 i}| \right| - \sum_{i=0}^{n_1} |c_{n_2 i} s_i| > \alpha^2. \end{aligned}$$

Similarly, choose n_3 greater than n_2 and such that

$$|c_{n_3 0}| + \dots + |c_{n_3 n_2}| < \alpha, \quad A_{n_3} > \alpha^6 + 3\alpha^3$$

* Whenever $c_{n_i i} = 0$ we understand that $|c_{n_i i}|/c_{n_i i}$ is to be replaced by unity here and in similar places below.

and put

$$s_i = \frac{1}{\alpha^3} \frac{|c_{n_3 i}|}{c_{n_3 i}} \quad (i = n_2 + 1, \dots, n_3).$$

Then it may be shown that $|t_{n_3}| > \alpha^3$. Proceeding in this way we obtain a sequence $s_0, s_1, s_2, \dots$, converging to zero, such that $\limsup_{n \rightarrow \infty} |t_n| = \infty$. Hence a third necessary condition for the regularity of our definition is that $\limsup_{n \rightarrow \infty} A_n$ shall be finite.

We have thus established the necessity of the conditions named in the following theorem:*

THEOREM I. *A necessary and sufficient condition that a sum shall be assigned to every convergent series (1) by the definition associated with (6) and that its value shall be the usual sum s of (1) is that*

- 1) $\lim_{n \rightarrow \infty} c_{nk} = 0$ for every k ;
- 2) $\lim_{n \rightarrow \infty} (c_{n0} + c_{n1} + \dots + c_{nn}) = 1$;
- 3) a number M , independent of n , shall exist such that for every n

$$|c_{n0}| + |c_{n1}| + \dots + |c_{nn}| < M.$$

It remains to prove the sufficiency of this condition. In view of 2), definition (6) and the fact that $\lim_{n \rightarrow \infty} (s_n - s)$ is zero when $\lim_{n \rightarrow \infty} s_n = s$ it is clearly sufficient to prove that $\lim_{n \rightarrow \infty} t_n$ exists and is zero when $\lim_{n \rightarrow \infty} s_n = 0$. Employing the last relation we see that for every positive ϵ there exists an N such that $|s_n| < \epsilon$ whenever $n > N$. But for $n > N$ we may write

$$t_n = (c_{n0}s_0 + \dots + c_{nN}s_N) + \sum_{i=N+1}^n c_{ni}s_i.$$

From 1) it follows that the quantity in parenthesis here approaches zero as n approaches infinity. In view of 3) the last sum is seen to be less than $M\epsilon$ in absolute value. Hence $\lim_{n \rightarrow \infty} t_n = 0$. This completes the proof of the theorem.

In the more usual definitions of summability by the method of mean values with finite reference the coefficients c_{ij} are

* The sufficiency of this condition has been proved by Silverman (l. c.). The necessity of the condition has been proved by Toeplitz (l. c.). The theorem has been generalized by Kojima (l. c.), who also extends the corollary. Less complete results of like import have been given by several writers. See, among others, the papers referred to near the end of §1.

non-negative real constants. For this case it should be observed that condition 3) is implied by condition 2).

An obvious modification of the proof of Theorem I leads to the following corollary:

COROLLARY. *A necessary and sufficient condition that $\lim_{n \rightarrow \infty} (c_{n0}s_0 + c_{n1}s_1 + \dots + c_{nn}s_n)$ shall exist and be finite in all cases in which $\lim_{n \rightarrow \infty} s_n$ exists and is finite is that*

- 1) $\lim_{n \rightarrow \infty} c_{nk}$ shall exist and be finite for every k ;
- 2) $\lim_{n \rightarrow \infty} (c_{n0} + c_{n1} + \dots + c_{nn})$ shall exist and be finite;
- 3) a quantity M , independent of n , shall exist such that

$$|c_{n0}| + |c_{n1}| + \dots + |c_{nn}| < M \text{ for every } n.$$

In view of relations (8') it is easy to show that the foregoing theorem may be put into the following equivalent form:

THEOREM II. *A necessary and sufficient condition that a sum shall be assigned to every converging series (1) by the definition associated with (7) and that its value shall be the usual sum s of (1) is that*

- 1) $\lim_{n \rightarrow \infty} a_{nk} = 1$ for every k ;
- 2) a constant M , independent of n , shall exist such that for every n

$$\sum_{i=0}^{n-1} |a_{n,i} - a_{n,i+1}| < M.$$

In case a_{nk} is positive and $a_{nk} - a_{n,k+1}$ is positive [negative] for every n and k it is clear that condition 2) is a consequence of condition 1).

It should be observed that the demonstration of Theorems I and II and the corollary to I requires no use of the hypothesis $c_{nn} \neq 0$, $a_{nn} \neq 0$.

It is an immediate consequence of Theorems I and II that Cesàro's definitions of sum are regular; and likewise that Knopp's and Chapman's extensions of them are regular. It is not difficult to establish similarly the regularity of Ford's extensions of Cesàro's definitions. In order to prove that Hölder's definition of order r is regular we observe that it is an immediate consequence of Theorem I that $\lim_{n \rightarrow \infty} s_n^{(r)} = s$ whenever $\lim_{n \rightarrow \infty} s_n^{(r-1)} = s$, and hence whenever $\lim_{n \rightarrow \infty} s_n^{(r-2)} = s$, $\dots$, and hence finally whenever $\lim_{n \rightarrow \infty} s_n^{(0)} = s$. From Theorem II it follows at once that de la Vallée Poussin's definition is regular. The regularity of other definitions by the

method of mean values with finite reference may be treated readily by the aid of Theorems I and II.

§3. *Mutual Consistency of Two Definitions of Sum by the Method of Mean Values with Finite Reference.*

We shall say that two definitions of sum of an infinite series are *mutually consistent** whenever it is true that the same sum is assigned by them to every series that is summable in accordance with both definitions.

Either one of the limits (see Hurwitz and Silverman, l. c., page 1)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=2}^n \left[1 + \frac{(-1)^{k+1}}{\log k} \right] s_k$$

affords a regular definition of the sum of a series (1). But the two definitions are not mutually consistent, since the first limit has the value 0 and the second the value 1 if $s_n = (-1)^{n+1} \log n$.

Let us consider the problem as to the mutual consistency of the definition associated with relation (6) and the following in which (1) is said to have the sum $T = \lim_{n \rightarrow \infty} T_n$, where

$$\begin{aligned} (18) \quad T_n &= r_{n0}s_0 + r_{n1}s_1 + \dots + r_{nn}s_n, & r_{nn} &\neq 0, \\ &= \alpha_{n0}u_0 + \alpha_{n1}u_1 + \dots + \alpha_{nn}u_n, & \alpha_{nn} &\neq 0. \end{aligned}$$

This fundamental problem relative to definitions by the method of mean values with finite reference seems not to have been resolved. In fact, so far as I am aware, it has received attention in a general way only in a single investigation, namely in the paper by Hurwitz and Silverman (already referred to), where the mutual consistency of all definitions of a certain subclass of these definitions has been established. We shall not reproduce the results of these authors.

Among the special cases of mutual consistency there is one of great importance which we shall treat further, namely, that in which every series which is summable to the sum s by a given one of the two given methods is also summable to the same sum s by the other of the two methods. In the preceding section we saw that Hölder's definitions are mutually

* Hurwitz and Silverman (l. c.) have used the term "consistent" in the sense of our term "mutually consistent."

consistent, the sum (Hr) existing and agreeing with the sum (Hk) for $k < r$ whenever the latter sum exists.

Let us consider the definitions of sum associated with relations (6) and (18) respectively. It is convenient to employ the infinite matrices

$$C = \begin{vmatrix} c_{00} & 0 & 0 & \cdots \\ c_{10} & c_{11} & 0 & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \quad \Gamma = \begin{vmatrix} \gamma_{00} & 0 & 0 & \cdots \\ \gamma_{10} & \gamma_{11} & 0 & \cdots \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Whenever conditions 1), 2), 3) of theorem I are satisfied we shall say that the matrix C is a *regular matrix of the first kind*; and likewise of course for other matrices of the same form.

In matrix notation we may write relations (6) and (18) in the respective forms

$$(t_n) = C(s_n), \quad (T_n) = \Gamma(s_n).$$

If we solve the first of these relations for the s 's, obtaining $(s_n) = C^{-1}(t_n)$, and substitute into the second, we have

$$(T_n) = \Gamma C^{-1}(t_n)$$

where ΓC^{-1} denotes the product of Γ and C^{-1} . Employing theorem I we have the following results (already essentially contained in Theorem I itself):

THEOREM III. *A necessary and sufficient condition that the definition associated with (18) shall assign a sum t to every series (1) to which the definition associated with (6) assigns the sum t is that the matrix ΓC^{-1} shall be a regular matrix of the first kind.*

This theorem may also be stated simply in terms of the matrices

$$A = \begin{vmatrix} a_{00} & 0 & 0 & \cdots \\ a_{10} & a_{11} & 0 & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \quad \alpha = \begin{vmatrix} \alpha_{00} & 0 & 0 & \cdots \\ \alpha_{10} & \alpha_{11} & 0 & \cdots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

We shall say that a matrix A is a *regular matrix of the second kind* whenever conditions 1) and 2) of Theorem II are satisfied.

We may write (7) in the form $(t_n) = A(s_n)$, whence we have $(s_n) = A^{-1}(t_n)$. In view of Theorem II we may now state the following result (already essentially contained in Theorem II itself):

THEOREM IV. *A necessary and sufficient condition that the definition associated with (18) shall assign a sum t to every series (1) to which the definition associated with (7) assigns the sum t is that the matrix αA^{-1} shall be a regular matrix of the second kind.*

It may be shown that Knopp (and hence Cesàro) summability of order r_1 assigns the sum s to any series to which this sum is assigned by the same summability of order r_2 where $r_1 > r_2 > -1$ (Knopp, l. c., page 5). A similar result holds also in the case of Chapman summability (Chapman, l. c.).

In the Paris *Comptes Rendus* for June, 1914, T. H. Gronwall and C. N. Moore independently proved that every series which is summable (Cr) to the sum s is summable to the same sum by the method of de la Vallée Poussin; and that there are series summable by the latter method but not by the method of Cesàro for any value of the order r .

§4. *Equivalence of Two Definitions of Sum by the Method of Mean Values with Finite Reference.*

We shall say that two definitions of sum of an infinite series are *equivalent* whenever it is true that every series which has a sum s in accordance with either of these definitions also has the same sum s in accordance with the other definition.

As a corollary from Theorems III and IV of the preceding section we now have immediately the following theorem:

THEOREM V. *A necessary and sufficient condition that the definitions of sum of an infinite series associated with (6) [or (7)] and (18), respectively, shall be equivalent may be put in either of the following two forms:*

- 1) *The matrices ΓC^{-1} and $C\Gamma^{-1}$ shall both be regular of the first kind;*
- 2) *The matrices αA^{-1} and $A\alpha^{-1}$ shall both be regular of the second kind.*

By means of this theorem Schur* has given a demonstration of the equivalence of summability (Cr) and summability (Hr) . That Hölder summability implies that of Cesàro was first proved by Knopp.† Independently, and by different methods, Schnee‡ and Ford§ showed that Cesàro summability implies

* *Math. Annalen*, vol. 74 (1913), pp. 447–458.

† Dissertation, Berlin, 1907.

‡ *Math. Annalen*, vol. 67 (1909), pp. 110–125.

§ *Amer. Journal of Mathematics*, vol. 32 (1910), pp. 315–326.

that of Hölder. More recently Faber* and Watanabe† have given other demonstrations of the equivalence of summability (Cr) and summability (Hr) .‡

Silverman (Dissertation, page 37) has established the equivalence of summability $(C1)$ and his φ -summability provided that φ_n approaches 1 monotonically as n increases indefinitely. He has pointed out (l. c., page 44) how this affords a convenient test for summability $(C1)$, and has indicated a means of obtaining an (unsatisfactory) extension of the method to the case of summability (Cr) .

§5. *Further Consideration of Definitions by the Method of Mean Values with Finite Reference.*

Besides the condition of being regular, already imposed upon our definitions of sum of divergent series, it is evidently desirable to restrict these definitions so that, as far as possible, the usual rules for operating with convergent series shall also be valid for summable divergent series.

Probably the first additional requirement to be demanded in the case of a given definition is that either of the series

$$(19) \quad u_0 + u_1 + u_2 + \dots,$$

$$(20) \quad u_1 + u_2 + \dots,$$

shall have a sum whenever the other has and that t_0 shall be equal to $t - u_0$ if t and t_0 are their respective sums.

Let us consider the general definition associated with (6) subject to the conditions of regularity as given in Theorem I. Let us write

$$t_n = c_{n0}s_0 + c_{n1}s_1 + \dots + c_{nn}s_n,$$

$$\bar{t}_n = c_{n0}s_1 + c_{n1}s_2 + \dots + c_{nn}s_{n+1}.$$

Then if (19) has the sum t we have $\lim_{n \rightarrow \infty} t_n = t$, while if (20) has the sum $t - u_0$ we have $\lim_{n \rightarrow \infty} \bar{t}_n = t - u_0$. Hence, a necessary and sufficient condition that our requirement shall be met is that either of the quantities $t_n, \bar{t}_n$ shall approach a (finite)

* *Münchener Sitzungsberichte*, 1913, pp. 519–531.

† *Tôhoku Mathematical Journal*, vol. 5 (1914), pp. 21–28.

‡ In this connection see also Landau's paper on the corresponding problem for integrals, *Leipziger Berichte*, vol. 65 (1913), pp. 131–138, and Fekete's paper on "absolute summability" by the methods of Hölder and Cesàro, *Math. és termész.*, vol. 32 (1914), pp. 389–425.

limit t whenever the other does; and hence that either of the quantities t_n' , $\bar{t}_n$ shall approach a finite limit t whenever the other does, where

$$t_n' = c_{n+1,1}s_1 + \cdots + c_{n+1,n+1}s_{n+1},$$

since $\lim_{n \rightarrow \infty} c_{n0} = 0$.

Symbolically, we may write $(t_n') = D(s_{n+1})$, where D denotes the infinite matrix

$$D = \begin{vmatrix} c_{11} & 0 & 0 & \cdots \\ c_{21} & c_{22} & 0 & \cdots \\ c_{31} & c_{32} & c_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Similarly, we may write $(t_n) = C(s_{n+1})$, where C denotes the matrix represented by this symbol in §3.

From considerations precisely like those involved in proving Theorem I we are now led to the following result:

THEOREM VI. *A necessary and sufficient condition that a regular definition of the form associated with (6) shall assign a sum to either of the series $u_0 + u_1 + u_2 + \cdots$ and $u_1 + u_2 + \cdots$ whenever it assigns a sum to the other and that t_0 shall be equal to $t - u_0$ where t and t_0 are respectively the sums of these series, is that each of the matrices DC^{-1} and CD^{-1} shall be regular of the first kind.*

A second natural requirement is the following: If a definition assigns sums t' and t'' to the first two of the series

$$(21) \quad \sum_{n=0}^{\infty} u_n, \quad \sum_{n=0}^{\infty} v_n, \quad \sum_{n=0}^{\infty} (u_n \pm v_n)$$

it shall assign the sum $t' \pm t''$ to the last of these series. This requirement is obviously met by all definitions by the method of mean values with finite reference.

Again, it is clear that such a definition assigns the sum kt to the series $ku_0 + ku_1 + \cdots$ whenever it assigns the sum t to the series $u_0 + u_1 + \cdots$.

So far as definitions by the method of mean values with finite reference are concerned we need to ascertain when the first alone of these additional requirements is met. For this we may utilize Theorem VI or we may proceed directly in the case of a particular definition. Thus it may be shown in

particular that this requirement is met by the definitions of Cesàro and Hölder.

Owing to the frequent use which has been made of Cesàro's definitions of sum it is desirable to refer to certain additional results associated with them.

THEOREM VII. *Necessary conditions in order that series (1) shall be summable (Cr) are that**

$$\lim_{n \rightarrow \infty} n^{-r} S_n^{(r-t)} = 0, \quad 0 < t \leq r; \quad \lim_{n \rightarrow \infty} n^{-r} u_n = 0.$$

From the definition of $S_n^{(r)}$ in §1 we have

$$S_n^{(r-1)} = S_n^{(r)} - S_{n-1}^{(r)}.$$

But $n^{-r} S_n^{(r)}$ approaches a finite limit l as n approaches infinity, since (1) is now summable (Cr) . Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-r} S_n^{(r-1)} &= \lim_{n \rightarrow \infty} \left[n^{-r} S_n^{(r)} - \left(\frac{n}{n-1} \right)^{-r} (n-1)^{-r} S_{n-1}^{(r)} \right] \\ &= l - l = 0. \end{aligned}$$

If we now assume that $\lim_{n \rightarrow \infty} n^{-r} S_n^{(r-t)} = 0$ for a given t we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-r} S_n^{(r-t-1)} \\ = \lim_{n \rightarrow \infty} \left[n^{-r} S_n^{(r-t)} - \left(\frac{n}{n-1} \right)^{-r} (n-1)^{-r} S_{n-1}^{(r-t)} \right] = 0. \end{aligned}$$

Induction now yields the first set of conditions. The last is similarly proved by means of the relations $s_n - s_{n-1} = u_n$ and $\lim_{n \rightarrow \infty} n^{-r} s_n = 0$.

From this theorem it follows that Cesàro's method is inapplicable if, to put it roughly, $|u_n|$ is *too large* when n is large. Hardy† has shown that the method is also inapplicable if $|u_n|$ is *too small* when n is large and in such wise that series (1) is not convergent. One of his theorems is as follows:

THEOREM VIII. *If $n |u_n| < K$, a quantity independent of n , then series (1) is not summable (Cr) for any value of r unless it is convergent.*

Reference may be made to certain additional results which

* Bromwich, *Infinite Series*, p. 318, obtains the last result in a different way.

† *Proc. London Math. Society*, ser. 2, vol. 8 (1909), pp. 301–320.

throw light upon the nature and the limitations of the methods of Cesàro, namely, those due to Fejer,* Hardy,† Chapman,‡ Landau,§ Kojima,|| Bohr,¶ Ottolenghi,** Hardy and Littlewood,†† Bromwich,‡‡ and Sannia.§§

§6. *Definitions of Sum by the Method of Mean Values with Infinite Reference.*

Let us now consider those definitions in which series (1) is said to have the sum t in case $\lim_{r \rightarrow \infty} t_r$ exists||| and has the finite value t , where

$$(22) \quad t_r = \sum_{k=0}^{\infty} c_{rk} s_k,$$

the c_{rk} being (real or complex) constants.

Again, we demand first of all that the definition shall be regular. In particular, we must have $\lim_{r \rightarrow \infty} t_r = 0$ when $s_k = 1$ and $s_i = 0$ for $i \neq k$. Hence $\lim_{r \rightarrow \infty} c_{rk} = 0$ for every k . Taking $s_k = 1$ for every k we see that for every permissible r the series $c_{r0} + c_{r1} + c_{r2} + \dots$ must converge and that

$$\lim_{r \rightarrow \infty} \sum_{k=0}^{\infty} c_{rk} = 1.$$

In order that t_r shall be defined for every convergent series and for each permissible value of r it is necessary (and sufficient) that $\lim_{n \rightarrow \infty} (c_{r0}s_0 + c_{r1}s_1 + \dots + c_{rn}s_n)$ shall exist and be finite in all cases in which $\lim_{n \rightarrow \infty} s_n$ exists and is finite. Thence, in view of condition 3) in the corollary to Theorem I, we see that the series $|c_{r0}| + |c_{r1}| + \dots$ must converge for every r .

* *Math. Annalen*, vol. 58 (1903), pp. 62–66.

† *Proc. London Math. Society*, ser. 2, vol. 6 (1908), pp. 255–264; ser. 2, vol. 8 (1909), pp. 301–320; *Math. Annalen*, vol. 64 (1907), pp. 77–94.

‡ *Proc. London Math. Society*, ser. 2, vol. 9 (1910), pp. 369–409.

§ *Prace matematyczno-fizyczne*, vol. 21 (1910), pp. 97–177.

|| *Tôhoku Mathematical Journal*, vol. 12 (1917), pp. 304–321.

¶ *Paris C. R.*, vol. 148 (1909), pp. 75–80; *Göttinger Nachrichten*, 1909, pp. 247–262.

** *Giornale di Matematiche*, vol. 49 (1911), pp. 233–279.

†† *Proc. London Math. Society*, ser. 2, vol. 11 (1912), pp. 1–16; *Palermo Rendiconti*, vol. 41 (1916), pp. 36–53.

‡‡ *Math. Annalen*, vol. 65 (1908), pp. 313–349.

§§ *Atti R. Accad. Sc. Torino*, vol. 50 (1915), pp. 97–112.

||| The limiting variable r may approach infinity over the set of numbers 1, 2, 3, . . . or over the continuum of real numbers. The treatment applies to both cases simultaneously.

If, in addition, a quantity M , independent of r , exists such that

$$\sum_{k=0}^{\infty} |c_{rk}| < M \text{ for every } r,$$

it is easy to show that the definition is regular. For suppose that $\lim_{n \rightarrow \infty} s_n = s$ and write $s_n = s + \epsilon_n$. Then we have

$$t_r = s \sum_{k=0}^{\infty} c_{rk} + \sum_{k=0}^{\infty} c_{rk} \epsilon_k;$$

whence it follows at once that $\lim_{r \rightarrow \infty} t_r = s$.

Thus we have the following result:

THEOREM IX. *A sufficient condition that the definition associated with (22) shall be regular is that*

- 1) $\lim_{r \rightarrow \infty} c_{rk} = 0$ for every k ;
- 2) *the series $c_{r0} + c_{r1} + \dots$ shall converge absolutely and its sum shall approach unity as r approaches infinity;*
- 3) *for every r the sum of the series $|c_{r0}| + |c_{r1}| + \dots$ shall be less than a quantity M independent of r .*

*The first two of these conditions are necessary for the regularity of the definition.**

In the case when the quantities c_{rk} are non-negative and real, conditions 1) and 2) are necessary and sufficient for the regularity of the definition.

Let us consider similarly the definition in accordance with which series (1) is said to have the sum t in case $\lim_{r \rightarrow \infty} t_r$ exists and has the finite value t , where

$$(23) \quad t_r = \sum_{k=0}^{\infty} a_{rk} u_k,$$

the a_{rk} being (real or complex) constants.

In order that the definition shall be regular it is necessary in the first place that $\lim_{r \rightarrow \infty} t_r$ shall be unity when $u_k = 1$ and $u_i = 0$ for $i \neq k$. Hence we must have $\lim_{r \rightarrow \infty} a_{rk} = 1$ for

* There is an obvious incompleteness about this theorem and the following one, presented here in the form in which I gave them in the lecture at Chicago. It was my intention to complete them before publication. But on the day following my lecture Professor T. H. Hildebrandt read a paper before the American Mathematical Society at Chicago giving the theorems in all their completeness and in elegant form, his results having been obtained before he knew of mine. Consequently I am leaving to him the duty of making known these desired results in their complete and satisfactory form.

every k . In what follows we assume that this condition is satisfied.

Again, in order that t_r shall be defined whenever (1) is convergent it is necessary (and sufficient) that the limit,

$$(24) \quad \lim_{n \rightarrow \infty} \sum_{k=0}^n a_{rk} u_k,$$

shall exist and be finite for every converging series (1). We may write

$$(25) \quad \begin{aligned} y_{rn} &= \sum_{k=0}^n a_{rk} u_k = \sum_{k=1}^n a_{rk} (s_k - s_{k-1}) + a_{r0} s_0 \\ &= \sum_{k=0}^{n-1} s_k (a_{rk} - a_{r, k+1}) + a_{rn} s_n. \end{aligned}$$

It follows now from the corollary to Theorem I that $\lim_{n \rightarrow \infty} y_{rn}$ will exist and be finite for every converging series (1) when and only when a constant M , independent of r , exists such that

$$|a_{rn}| + \sum_{k=1}^n |a_{rk} - a_{r, k+1}| < M \text{ for every } r;$$

and hence when and only when the series*

$$(26) \quad \sum_{k=1}^{\infty} |a_{rk} - a_{r, k+1}|$$

is convergent. This implies, in particular, that $\lim_{n \rightarrow \infty} a_{rn}$ exists and is finite.

If in addition to the foregoing limitations on the coefficients a_r , they are further restricted by the requirement that a constant M_1 shall exist, independent of r , such that the sum of the series in (26) shall be less than M_1 for every r , it is easy to show that our definition is regular. For, in view of (25) we have

$$\begin{aligned} t &= \lim_{r \rightarrow \infty} t_r = \lim_{r \rightarrow \infty} \left[\sum_{k=0}^{\infty} s_k (a_{rk} - a_{r, k+1}) + s \lim_{n \rightarrow \infty} a_{rn} \right] \\ &= \lim_{r \rightarrow \infty} \left[\sum_{k=0}^{\infty} s (a_{rk} - a_{r, k+1}) + s \lim_{n \rightarrow \infty} a_{rn} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} e_k (a_{rk} - a_{r, k+1}) \right], \end{aligned}$$

* This result is obtained by Kojima, *l. c.*, p. 305.

where $s + \epsilon_k$ is written for s_k . Hence

$$t = s + \lim_{r=\infty} \sum_{k=0}^{\infty} \epsilon_k (a_{rk} - a_{r, k+1}),$$

whence it follows at once that $t = s$.

Hence we have the following theorem:*

THEOREM X. *A sufficient condition that the definition associated with (23) shall be regular is that*

- 1) $\lim_{r=\infty} a_{rk} = 1$ for every k ;
- 2) the series $|a_{r0} - a_{r1}| + |a_{r1} - a_{r2}| + \dots$ shall converge;
- 3) a constant M , independent of r , shall exist such that the sum of the series in 2) shall be less than M for every r .

Moreover, the first two of these conditions are necessary.†

In the case when the quantities $a_{rk} - a_{r, k+1}$ are all real and not negative, or all real and not positive, it is obvious that condition 3) is implied by conditions 1) and 2); and hence in either of the conditions named 1) and 2) are necessary and sufficient for the regularity of the definition.

The regularity of Borel's exponential definition follows at once from Theorem IX. For, we have

$$c_{rk} = \frac{r^k}{k!} e^{-r},$$

so that conditions 1), 2), 3) of the theorem are satisfied.‡

By writing $rx = r - 1$ LeRoy's definition (15) may be thrown into the form

$$t = \lim_{x=1-0} \sum_{k=0}^{\infty} \frac{\Gamma(kx + 1)}{\Gamma(k + 1)} u_k = \lim_{r=\infty} \sum_{k=0}^{\infty} \frac{\Gamma(k - k/r + 1)}{\Gamma(k + 1)} u_k.$$

If we put

$$a_{rk} = \frac{\Gamma(k - k/r + 1)}{\Gamma(k + 1)} = \left(1 - \frac{1}{r}\right) \frac{\Gamma(k - k/r)}{\Gamma(k)}$$

and employ the asymptotic properties of the gamma function

* Certain related but different results are due to C. N. Moore, *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 299–330. The same author has also given similar theorems for double series, *ibid.*, vol. 14 (1913), pp. 73–104.

† See the footnote to the preceding theorem.

‡ For additional theorems concerning Borel's exponential method of summation see Hardy and Littlewood, *Palermo Rendiconti*, vol. 41 (1916), pp. 36–53.

it is easy to show that the conditions of Theorem X are satisfied and hence that the definition in consideration is regular.

According to Borel's integral method of summation we take for the sum of (1) the quantity,

$$t = \int_0^\infty e^{-a} \left(\sum_{k=0}^\infty \frac{a^k}{k!} u_k \right) da,$$

where it is assumed that series (1) is such that the infinite series appearing in this expression converges for every value of a . This may be written in the form

$$\begin{aligned} t &= \lim_{r \rightarrow \infty} \int_0^r e^{-a} \left(\sum_{k=0}^\infty \frac{a^k}{k!} u_k \right) da \\ &= \lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{k=0}^n \left[\int_0^r e^{-a} \frac{a^k}{k!} da \right] u_k. \end{aligned}$$

Hence, if we take

$$a_{rk} = \int_0^r e^{-a} \frac{a^k}{k!} da,$$

we exhibit the Borel integral definition as an element of the class of definitions by the method of mean values with infinite reference. Since

$$a_{rk} - a_{r, k+1} = e^{-r} \frac{r^{k+1}}{(k+1)!},$$

as one sees readily through integrating by parts the integral defining $a_{r, k+1}$, it is easy to show that the hypotheses of theorem X are satisfied; and hence that the Borel integral definition is regular.

A comparison of Theorems IX and X with Theorems I and II brings out the fact (already alluded to in the footnotes) that the former are not in altogether the same satisfactory form as the latter; they are completed by the (as yet unpublished) work of Hildebrandt. So far as I am aware no general theorems yet exist concerning the mutual consistency of definitions by the method of mean values with infinite reference nor concerning the equivalence of such definitions (see related matters in §§3 and 4). We may propose these as two fundamental problems, or perhaps as two aspects of the same fundamental problem, in the theory of summable series. Other considerations analogous to those of §5 still await de-

velopment for the case of definitions by the method of mean values with infinite reference.

§7. *Definitions of Sum by Means of Integrals.*

The theory of Borel's definition of sum by means of an integral has been so well expounded in Borel's *Séries divergentes* and Bromwich's *Infinite Series* that we need to give here only a brief summary of the main results.* We confine attention to the case of absolute summability.†

If one has a polynomial function $P(u, v, w, \dots)$ of a finite number of variables $u, v, w, \dots$, the coefficients being numerical, and if one replaces $u, v, w, \dots$ by absolutely summable series and combines the result according to the usual rules of operating with convergent series one obtains an absolutely summable series whose sum is equal to the numerical value obtained by replacing $u, v, w, \dots$ in P by the sums of the corresponding series (Borel, l. c., page 108).‡

Denote by u, v, w power series

$$u = \sum_{n=0}^{\infty} u_n x^n, \quad v = \sum_{n=0}^{\infty} v_n x^n, \quad w = \sum_{n=0}^{\infty} w_n x^n,$$

and suppose that these are absolutely summable for $x = x_0$. Let

$$P(u, v, w, u', v', w', \dots, u^{(\lambda)}, v^{(\lambda)}, w^{(\lambda)}, x)$$

denote a polynomial in u, v, w and their derivatives up to order λ inclusive, the coefficients of which are series in integral powers of x whose radius of convergence is greater than $|x_0|$. If in P one replaces u, v, w by the corresponding series and performs the indicated operations as if the series were convergent, one obtains a series S which is absolutely summable for each value of x on the straight line from 0 to x_0 (exclusive of x_0 itself) and which defines an analytic function F which is regular in the interior of a circle of which the line joining 0 and x_0 is a diameter. This analytic function is precisely that

* In the preceding section we saw that Borel's definition of summability (but not of absolute summability) is regular.

† Borel's statement (l. c., p. 100) that every convergent series is absolutely summable is incorrect, as has been pointed out by Hardy, *Quar. Journal of Math.*, vol. 35 (1903), pp. 25–28. But every absolutely convergent series is absolutely summable.

‡ Related results for summable series (not absolutely summable) are given by Hardy, l. c., p. 43.

which P becomes when in P one replaces u, v, w not by the series but by the corresponding analytic functions. Moreover, F is identically zero when and only when the series S has all its coefficients zero, that is to say, when and only when the series u, v, w formally satisfy the relation

$$P(u, v, w, u', \dots, w^{(\lambda)}, x) = 0.$$

In this case the corresponding analytic functions u, v, w also satisfy this relation (Borel, l. c., page 114).

If the foregoing theorem is applied to the case of the differential equation

$$f(x, y, y', \dots, y^{(\lambda)}) = 0,$$

where f is analytic in x at $x = 0$ and algebraic in y and its derivatives, it results at once that, if an absolutely summable series y formally satisfies the differential equation, the analytic function defined by the Borel integral sum of this series is a solution of the differential equation (Borel, l. c., page 115).

Suppose that the power series

$$\varphi(x) = u_0 + u_1x + u_2x^2 + \dots$$

has a finite (non-zero) radius of convergence. Let l denote a line through a singularity of the function defined by the power series and perpendicular to the radius vector from zero to this singularity. Let P denote the "polygon" which contains in its interior the point zero and extends from zero in every given direction to the nearest of these lines l in such direction. Then the series $\varphi(x)$ is absolutely summable for points in the interior of P ; it is not absolutely summable at any point exterior to P ; on the boundary of P it may or may not be absolutely summable (Borel, l. c., page 128).*

Through use of the foregoing result and tests for the absolute summability of series one is able to treat certain phases of the problem as to the position of the singularities of a function defined by a power series (Borel, l. c., page 136).*

Borel's integral method of summation is thus seen to yield an important contribution (among others) to the fundamental

* The method of analytic continuation suggested by the results in these two paragraphs may be extended by the use of a definition of summability based on the function $\varphi(a) = e^{a^k}$ instead of on the function $\varphi(a) = e^a$ (Borel, l. c., p. 129 ff.). The "polygon" of summability is replaced by a certain curvilinear figure. See also a paper by E. Lindelöf, *Bull. Soc. Math. France*, vol. 29 (1901), pp. 157-160.

problem of deducing the properties of an analytic function from the properties of its Taylor development.* Contributions to the study of this problem have been made by many authors from diverse points of view. One of the most important of these in connection with the subject of summability is a memoir by LeRoy.†

Let us consider the divergent series

$$\sum_{n=0}^{\infty} \alpha_n z^n.$$

Put

$$\alpha_n = \Gamma(pn + 1) a_n$$

and denote by $F(z)$ the function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Suppose that the radius of the circle of convergence of the last series is finite (and different from zero) and that the circle of convergence is not a natural boundary of the function defined by it. Then for the sum $f(z)$ of the first series LeRoy (l. c., page 416) writes

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n = \frac{1}{p} \int_0^{\infty} e^{-x^{1/p}} x^{-1+1/p} F(zx) dx,$$

provided that the last integral is convergent. For $p = 1$ this reduces to Borel's definition. Under appropriate broad conditions such summable divergent series as are here introduced are amenable to the usual methods of computation employed in the case of convergent series. The theory has applications to differential equations and analytic continuation of the same essential character as those of the special case in which Borel's definition is sufficiently far-reaching. We must refer the reader to LeRoy's memoir for a fuller account of his important investigations in connection with this and other

* Conversely, one may think of any method of analytic continuation as a method for the summation of divergent series (Borel, l. c., p. 120). Note how LeRoy (see next footnote) has been led in this way to new definitions of sum. See especially pp. 405 ff. of his paper.

† *Toulouse Annales*, ser. 2, vol. 2 (1900), pp. 317-430. See also the work of Servant (referred to at the end of §1 above).

definitions of summability. See also his notes in *Paris Comptes Rendus* for 1898, 1899, 1900.*

Reference may also be made to extensions of Borel's method by Cunningham† and by Buhl.‡ Compare also the method of Barnes (see reference near end of §1).

§8. *General Requirements Which Should be Met in Any Definition of Sum of an Infinite Series.*

In §5 we have considered some general requirements which should be met by every definition of sum of an infinite series. It appears that a great many definitions, not all mutually consistent, meet all these requirements. In view of this fact and with a desire to introduce greater uniformity into the general theory of summability, some authors have felt that certain further general restrictions are distinctly to be desired. Thus W. O. Mendenhall§ and W. B. Ford|| have insisted on the desirability of imposing the so-called boundary value condition. Thus these authors propose to confine attention to those series (1) for which the corresponding power series

$$f(x) = \sum_{n=0}^{\infty} u_n x^n$$

has a radius of convergence equal to unity¶ and then to agree to retain those definitions alone for which the sum s of (1), when existent, satisfies the relation**

* Besides the references already given see also Ricotti, *Giornale de Matematiche*, vol. 48 (1910), pp. 79–111, for a treatment of the methods of Borel and LeRoy; Maillet, *Annales École Norm. Sup.*, ser. 3, vol. 20 (1903), pp. 487–518; Van Vleck, *Boston Colloquium*, 1903, pp. 92–107.

† *Proc. London Math. Society*, ser. 2, vol. 3 (1905), pp. 157–169.

‡ *Bulletin des Sciences mathématiques*, vol. 42 (1907), pp. 340–346.

§ Michigan dissertation, 1911. This dissertation has not been published. Through the kind response of the author to a request of mine I have had the opportunity to examine a manuscript copy of it. Its more important results are to be found in Ford's book on divergent series.

|| Ford, *Studies on Divergent Series and Summability*, pp. 82 ff. Also in his address (December, 1917) as retiring chairman of the Chicago Section of the American Mathematical Society, this *BULLETIN*, vol. 25 (1918), pp. 1–15.

¶ The restriction to such series appears to be particularly unfortunate in view of the important applications of the theory of summability to power series with zero radius of convergence.

** The demand here essentially is that we shall restrict attention to certain of those definitions of sum which are mutually consistent with a single given definition, namely, that in which series (1) is said to have the

$$s = \lim_{x \rightarrow 1-0} f(x).$$

An earlier treatment of the general considerations involved in a restriction of this sort is to be found in LeRoy's memoir (cited in §7). He takes a more general point of view than Mendenhall and Ford in that he allows the radius of convergence of the power series to be less than 1 and considers the question of assigning to series (1) the sum $f(1)$, where an element of $f(x)$ is defined as above. He also discusses the problem for the case when the radius of convergence of the power series is zero.

In view of the fact that convergent power series are a much restricted class of infinite series and that the theory of summability has made significant and important conquests in the theory of expansions in orthogonal functions (including Fourier series), in the theory of Dirichlet series and factorial series, and promises important applications to the more general expansion problems arising in the theory of difference equations, such a restriction as that desired by Mendenhall and Ford should be insisted upon only for the most cogent reasons. Furthermore one of the far-reaching applications of the theory of summability is to the case of descending power series which diverge for every value of the variable, an application which leads to consequences of large importance in the theory of differential and difference equations. Hence this class of series must not be ruled out; moreover, it is difficult to see why a definition which is to be used in the case of such series must of necessity satisfy the boundary value condition. Thus it appears to me that no satisfactory reasons have been advanced for this restriction proposed by Mendenhall and Ford and consequently that there is not yet any sufficient ground for confining attention to definitions satisfying the boundary value condition. Those which do satisfy it doubtless form an important class having a well-defined usefulness in the study of analytic functions; but there seems not yet to be any valid reason for supposing that other classes are not also important.

sum s when the limit

$$\lim_{x \rightarrow 1-0} (u_0 + u_1x + u_2x^2 + \dots)$$

exists and has the finite value s ; and that we shall reject every definition which assigns a sum to a series (1) to which this last definition does not assign a sum.

For the case of only a few definitions has it yet been determined whether the so-called boundary value condition is satisfied. For these consult the work of Mendenhall and Ford.

The other extreme as regards the matter of limitation upon the definition of summability or freedom in this respect has not failed to be represented in the literature. Thus Hardy* has formulated a principle which Bromwich (*Infinite Series*, pages 267-8) states in the following words: "If two limiting processes, performed in a definite order on a function of two variables, lead to a definite value X , but, when performed in the reverse order, lead to a meaningless expression Y , we may agree to interpret Y as meaning X ." The property implied in this principle certainly belongs to many of the current definitions of sum; but it does not appear to have been a useful guide in the formulation of any of the important special definitions. It seems to allow too great a variety of possibilities to provide anything of marked value in the way of specific definitions. It is useful, however, as a unifying principle.

§9. *Applications to the Theory of Dirichlet Series.*

From Theorem VII it follows that Cesàro's methods of summation are limited in a way which forbids their application to the problem of the analytical continuation of a function defined by a power series; but they have been of the greatest use in the study of the function on the circle of convergence of the power series which represents it. Owing to the delicate character of the convergence of a Dirichlet series it is natural to suppose that Cesàro's methods would find wider application here than in the theory of power series whose character of convergence is much more crude. The first applications of this sort were made independently by Bohr and M. Riesz, who showed that the region of Cesàro summability of a Dirichlet series may be greater than its region of convergence, so that there exist regions of summability in which the series diverges while, nevertheless, Cesàro's means afford the ana-

* *Quar. Journal Math.*, vol. 35 (1904), pp. 22-47. See also Hardy and Chapman, *ibid.*, vol. 42 (1911), pp. 181-215; Chapman, *ibid.*, vol. 43 (1911), pp. 1-52. In the last paper Chapman treats a generalization of Hardy's principle.

lytical continuation of the function defined by the series within its region of convergence.*

Although Cesàro's methods thus have an important application in the theory of Dirichlet series it appeared from the investigations of Riesz that a certain disadvantage still existed which might be overcome by the introduction of a new type of definition which should maintain certain features belonging to the definitions of Cesàro. Thus Riesz was led to introduce his definition by "typical means," as given in §1 in connection with relation (12).

In the case when $\lambda_n = n$ it has been shown by Riesz† that his definitions, for varying r , are equivalent to those of Cesàro, Knopp, and Chapman, when in each case the same value of r is taken.

An excellent account of this method of summation and of its application to the theory of Dirichlet series has been given by Hardy and Riesz.‡ For this reason we shall not attempt an exposition of the matter. We state merely a few of the leading properties of the definition as developed by these authors.

The definitions of Riesz are regular. If (1) is summable $(R\lambda r)$ to the sum s , it is summable $(R\lambda r')$ to the same sum s , for every r' greater than r . If (1) is summable (Rlr) , where $l_n = e^{\lambda n}$, then it is summable $(R\lambda r)$ to the same sum. If (1) is summable $(R\lambda r)$, where $\lambda_n = n$, it is also summable $(R\lambda r)$ to the same sum where $\lambda_n = \log n$. If μ is any logarithmico-exponential function of λ , and if (1) is summable $(R\lambda r)$ then it is summable $(R\mu r)$. Speaking roughly, we may say (in view of the last result) that the efficacy of the method $(R\lambda r)$ increases as the rate of increase of the function λ decreases.§

* Borel's exponential method of summation is also applicable to Dirichlet series; see Hardy, *Proc. London Math. Society*, ser. 2, vol. 8 (1909), pp. 277–294. See also Hardy's paper on the so-called Abel's method of summation as applied to Dirichlet series, *Quar. Journal of Math.* vol. 47 (1916), pp. 176–192.

† Paris *Comptes Rendus*, vol. 152 (1911), pp. 1651–1654.

‡ "The General Theory of Dirichlet Series," Cambridge University Press, 1915. This monograph should also be consulted for treatment of other matters relating to the summability of Dirichlet series.

§ We may refer to the following papers which appeared later than the Hardy-Riesz monograph:

Nalli, *Palermo Rendiconti*, vol. 40 (1915), pp. 44–70; 42 (1917), pp. 61–72.

Hardy, Paris *Comptes Rendus*, vol. 162 (1916), pp. 463–466.

Hardy, *Quar. Journal Math.*, vol. 47 (1916), pp. 176–192.

Hardy, *Proc. London Math. Society*, ser. 2, vol. 15 (1916), pp. 72–88.

§10. *Applications to the Expansion Problems Arising in the Theory of Difference Equations.*

This account of the general aspects of the theory of summability we shall bring to a close by indicating briefly an important application the development of which lies mostly in the future. In several memoirs* which I have presented to the Society in the last two or three years I have given an outline of the main features of what appears to me to be the central expansion problem in the theory of difference equations. This problem has to do with series of the form†

$$\Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}$$

where the quantities c_n are independent of x and the function $g(x)$ is restricted mainly as to its analytic and asymptotic character in a certain sector V including in its interior the positive axis of reals. The asymptotic form is as follows:

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right),$$

where $P(x)$ and $Q(x)$ are polynomials and the a 's are constants. It is assumed that $g(x)$ is analytic in V when $|x|$ is sufficiently large but finite.

The most important cases (and the ones arising most naturally from the theory of linear difference equations) are those in which $P(x)$ is a linear function of x and $Q(x)$ is a linear function or a constant. Certain special functions $g(x)$ belonging to the class thus defined give rise (see memoir II) to factorial series and to some generalizations of them which play the fundamental rôle in certain important recent investigations (see references in memoir II).

* These memoirs will be referred to by the numbers in the following list:

I. *Transactions Amer. Math. Society*, vol. 17 (1916), pp. 207-232.

II. *Bulletin Amer. Math. Society*, vol. 23 (1917), pp. 407-425.

III. *Amer. Journal of Math.*, vol. 39 (1917), pp. 385-403.

IV. *Amer. Journal of Math.*, vol. 40 (1918), pp. 113-126.

† A similar problem arises for series of the form

$$\sum_{n=0}^{\infty} c_n \frac{g(nx)}{g(x)},$$

where $g(x)$ has asymptotic properties similar to those defined in the text. These series are now being investigated by L. L. Steimley, a student in the University of Illinois.

So far the question of summability of series $\Omega(x)$ has been treated only for the case of factorial series; and the methods of Cesàro alone have been used for this purpose. Landau* has shown that the factorial series and the Dirichlet series

$$\sum_{s=0}^{\infty} \frac{a_s s!}{x(x+1) \cdots (x+s)}, \quad \sum_{s=1}^{\infty} \frac{a_s}{s^x}$$

have the same points of convergence (except for $x = 0, -1, -2, \dots$). The interior of the region of convergence of each series is the portion of a plane to the right of a line $R(x) = \lambda$, where λ is an appropriately determined constant and $R(x)$ stands for the real part of x . Bohr† has established the existence of a sequence of real numbers $\lambda_1, \lambda_2, \lambda_3, \dots$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$) such that each of the two foregoing series is summable (Cr) in a half-plane $R(x) > \lambda_r$ ($r = 1, 2, 3, \dots$) but not for $R(x) < \lambda_r$. As n approaches infinity λ_n approaches a limit $\bar{\lambda}$. Bohr (l. c.) has shown that the line $R(x) = \bar{\lambda}$ plays a fundamental rôle for functions defined by the Dirichlet series. Nörlund‡ has pointed out that the number $\bar{\lambda}$ does not enjoy a similar property for the factorial series; this raises the question as to whether other methods of summation may do for the factorial series what the methods of Cesàro do for Dirichlet series.

The function $f(x)$ defined by the foregoing factorial series has an asymptotic representation§ (in general divergent) of the form

$$f(x) = \frac{A_0}{x} + \frac{A_1}{x^2} + \frac{A_2}{x^3} + \dots$$

This series is summable by the exponential method of Borel||

* *Münchener Sitzungsberichte*, vol. 36 (1906), pp. 151–218.

† *Göttinger Nachrichten*, 1904, pp. 247–262. Bohr states also several important properties of the regions of summability of different orders. He also derives similar results for the case of series of binomial coefficients.

‡ *Paris Comptes Rendus*, vol. 158 (1914), pp. 1325–1328.

§ See Nielsen, *Annales Ecole Norm. Sup.*, ser. 3, vol. 2 (1904), pp. 449–458. See also the treatment of a more general problem in my memoirs III and IV.

|| Nörlund, l. c.; *Paris Comptes Rendus*, vol. 158 (1914), pp. 1252–1253; *Acta Mathematica*, vol. 37 (1914), pp. 327–387. A certain natural extension of factorial series is treated here, the theory still being contained in that of the general series $\Omega(x)$.

Compare also in this connection, Pincherle, *R. Accad. L. Rend.*, ser. 5, vol. 13 (1904), pp. 513–519, and my memoir IV already cited.

and the sum obtained by this method is $f(x)$:

$$f(x) = \int_0^{\infty} e^{-xt} F(t) dt,$$

where

$$F(t) = A_0 + \frac{A_1}{1!} x + \frac{A_2}{2!} x^2 + \dots$$

A number l exists such that this integral converges when $R(x) > l$ and diverges when $R(x) < l$. This number l plays a fundamental rôle as regards the properties of the function $f(x)$ defined by the factorial series.

We have here in a special case two aspects of the general theory of summability of the series $\Omega(x)$, a theory of importance the development of which will lead to significant extensions of our knowledge of one of the most fundamental expansion problems in analysis. Early in 1917, Mr. Charles F. Green, a student at the University of Illinois, was beginning work upon this subject, looking towards a doctor's dissertation; but his labor has been interrupted by more pressing duties and he is now engaged as a pilot in the aviation service with the American Army in France.

UNIVERSITY OF ILLINOIS.

ON THE PROBLEM OF THE RESISTANCE INTEGRAL.

BY PROFESSOR TSURUICHI HAYASHI.

THE problem of minimizing the resistance integral seems to be of three main varieties.

1. Newton's problem:*

To get a solid of revolution formed by revolving a curve passing through two given points about an axis which shall experience a minimum resistance when it moves through a fluid in the direction of its axis.

The solution is the well-known transcendental curve.

* *Philosophiæ Naturalis Principia Mathematica*, 1687, Book 2, Section 7, Prop. 34, Scholium.

2. Tarleton's problem:*

To get a solid of revolution on a given base with a given volume which shall experience a minimum resistance when it moves through a fluid in the direction of its axis.

The solution is a hypocycloid of three cusps.†

3. Euler's problem:‡

The full enunciation of this problem is found in Professor G. H. Light's paper in the July number of this BULLETIN, 1918, volume 24, page 480. The solution can be easily obtained in its parametrical form, which is quite the same as that for Tarleton's problem. So without using Professor Light's method, we may use Todhunter's method to show that the solution is a hypocycloid of three cusps. Todhunter's method depends on the relation

$$dy/dx = p = \tan \phi,$$

and seems to be more easy and more natural.

August, 1918.

NOTE ON EDITIONS OF VON STAUDT'S GEOMETRIE DER LAGE.

BY PROFESSOR R. C. ARCHIBALD

(1) The first editions of K. G. C. von Staudt's *Geometrie der Lage* (G.) and *Beiträge zur Geometrie der Lage* (B.) were issued in 4 parts, paper covers, by "Verlag von Bauer und Raspe (Julius Merz)," Nürnberg, 1847-1860, § each of the four parts having a separate title page.

(2) An Italian translation of (G.), and of some paragraphs of (B.), was published in 1889.||

* *Philosophical Magazine*, 4th series, vol. 34 (1867).

† See Todhunter's *Researches in the Calculus of Variations*, 1871, pp. 196-198. It is reproduced in Carll's *Calculus of Variations*, 1881, pp. 144-146.

‡ *Scientia Navalis*, Prop. 53.

§ (G.): 1847 (on the reverse of the title page is: "Erlangen, gedruckt bei J. J. Barfus," pp. 6 + 216). (B.): 1856 ("Druck von Junge & Sohn in Erlangen," pp. 6 + 130), 1857 ("Druck von Junge & Sohn in Erlangen," pp. 131-284), 1860 ("Druck der A. E. Junge'schen Universitätsbuchdruckerei," pp. 6 + 285-396).

|| *Geometria di Posizione di Giorgio Carlo Cristiano v. Staudt. Traduzione dal tedesco a cura del dott. M. Pieri, professore all'Accademia militare, preceduta da uno studio del Prof. C. Segre sulla vita e le opere del v. Staudt.* Torino, Bocca, 28 + 233 pp.

These editions are well known. There was, however, (3) another German edition, which seems entirely to have escaped the notice of the bibliographer and historian.* A copy of this edition has been recently presented to Brown University. In its title pages are lacking for each of the three parts of (B.). While the title page of (G.) is without date, the place of publication is given: "Nürnberg, Verlag der Fr. Korn'schen Buchhandlung."†

The total number of pages in each of the four parts is the same as in (1), and the "Berichtigungen," on pages 216 (G.) and 283 (B.) are reprinted without corresponding changes in the text. While there is in general a one-to-one correspondence in the content of pages numbered the same in (1) and (3), this is not absolute; for example, in (G.): pages III, 9, and 203 (where a correction of the original has been made); and in (B.): pages 15, 47, 134-5, and 187. The print page of the new edition is slightly longer and narrower than that of the old one.

The only clue I could find as to the date of this edition was in the British Museum catalogue, volume SQ.—States of the Church, London, 1896) which lists (G.) as published in "Nürnberg, [1883]." On appealing to the Museum for its authority in determining the date "[1883]," Mr. Alfred W. Pollard, the erudite keeper in the Library wrote in part as follows: "Our copy of K. G. C. von Staudt's Geometrie der Lage (Nürnberg, Verlag der Fr. Korn'schen Buchhandlung) cannot have been published later than 1883, as we paid for it on 6 July of that year, the invoice being dated 20 January. A young cataloguer (he came that June) cataloguing it in October assumed that it was a new book. I think it was really published between 1877 and 1880. On the cover it advertises an Atlascommentar of G. Wenz which I find in Heinsius dated 1877.‡ It also advertises Dr. L. Wöckel's Geometrie der Alten in einer Sammlung von 850 Aufgaben. This number 850 occurs in the eighth and ninth editions (published by Bauer and Raspe 1869 and 1871) and in the tenth and eleventh (published by

* For example: Poggendorff, Kayser (Bücher-Lexicon), M. Cantor (article on von Staudt in Allgemeine deutsche Biographie, vol. 35, 1893), and Encyclopédie (III. 1. 2. p. 207, 1915) refer to only the one edition of (G.).

† On the reverse side is "Druck von E. Th. Jacob in Erlangen." It will be seen later (in connection with Wöckel's Geometrie der Alten) that Korn was probably the successor of Bauer and Raspe.

‡ In "Kayser" this date is 1876, which is, according to its catalogue, the date of the copy in the Library of Congress.

Korn, 1873 and 1876).† In the twelfth edition (published by Korn in 1880 it was increased to 856. On the supposition that Korn kept his advertisements up to date, von Staudt's book must have appeared after Wenz's *Atlascommentar* of 1877* and before the twelfth edition of Wöckel's *Geometrie der Alten* in 1880. Its probable date is thus a little before 1880."

Judging by typography, the new editions of (G.) and (B.) were published about the same time.

An entry in the "Katalog des mathematischen Lesezimmers der Universität Göttingen"† suggests that there may have been yet a third German edition of (G.) in 1846, since it has "Nürnberg [1846]" after the title. More probably however it is the edition described above as without date, incorrectly catalogued.

MATHEMATICAL PERIODICALS.

Union List of Mathematical Periodicals. By D. E. SMITH and C. E. SEELY. (Bureau of Education, Bulletin 1917, No. 9.) Washington, D. C., 1918. 60 pages. Price 10 cents.

THIS union list is a guide to the location of certain periodicals. The compilers have stated: it was "prepared for the use of research students in mathematics in the universities of the United States. It is not intended to be a complete list of all publications of this kind; indeed, such a catalogue, while very desirable from the bibliographical and historical standpoint, would not ordinarily serve the purposes of the graduate student in mathematics as well as a brief list of this nature. The selection has been made after consultation with a number of professors in those universities that have most to do with directing research work in mathematics in this country, and it represents the periodicals which, in the judgment of these advisers, the students will be most apt to consult in his investigations."

Of about 165 serial publications so selected, somewhat less than one half are wholly mathematical. In connection with

* The number "850" occurs in the title from the fourth (1856) to the eleventh editions inclusive.

† Bearbeitet von K. Hiemenz. Mit einem Vorwort von F. Klein. Leipzig, 1907, p. 92.

the title of each periodical are given: (1) the series and volume numbers, with years and place or places of publication. In case some society or academy is publisher, this is also indicated; (2) an alphabetical list of certain libraries owning one or more of the volumes, and an exact statement of the volumes to be found in such libraries.

The 52 libraries considered are situated in the following 21 states: California, Connecticut, District of Columbia, Idaho, Illinois, Indiana, Iowa, Maryland, Massachusetts, Michigan, Minnesota, Missouri, Nebraska, New Jersey, New York, Ohio, Pennsylvania, Rhode Island, Texas, Virginia, and Washington.

In the pamphlet are two main divisions headed "Mathematical Periodicals" and "Periodicals Partly Mathematical," and a detailed index. The arrangement of this whole is convenient for ready consultation although an arrangement according to countries (as in Greenstreet's list mentioned below) would have been more compact. There can be no doubt but that the list will be of great service to many students.

Typical of comments, the number of which might be much extended, are the following: Among American periodicals listed *The Mathematical Visitor* (1878-1895), *The Mathematical Review* (1896-1897) and *Bryn Mawr College Monographs, Contributions from the Mathematical and Physical Departments* (1904, 1909) are conspicuous by their absence. While *Journal de Mathématiques élémentaires* (Vuibert) is considered, the much more important *Journal de Mathématiques élémentaires (et spéciales)* (Bourget, de Longchamps, etc., 1877-1901) is not. *Nouvelle Correspondance mathématique* (1875-1880) is listed but the more substantial *Correspondance mathématique et physique* (Quetelet, 1825-1839) as well as the most valuable *Correspondance sur l'Ecole polytechnique* (1804-1816) are not. Since continuations are sometimes indicated, why was it not also noted that *Nouvelle Correspondance mathématique* was continued as *Mathesis*, and *The Analyst* as the *Annals of Mathematics*? There is no reference to the immediate predecessors of *Journal of the Indian Mathematical Club* and of Conti's *Il Bolletino di Matematica* whose early volumes, at least, were published at Bologna and not at "Roma (Rome)." It is surely misleading to refer to the *Ladies' Diary*, 1704-1773, as "reprinted in 1775." Without doubt the editors had in mind Hutton's edition issued in 14 numbers 1771-1775, which

may hardly be accurately termed a "reprint." To be consistently complete in this connection *The Diarian Repository* (1771-1774), and T. Leybourne's Questions from *The Ladies' Diary* (1817), should have been mentioned also. In the light of the quotation made above it is not clear how Dodgson's Mathematical Repository (which is not a periodical and has nothing to do with Leybourn's serial, page 21), *The Ladies' Diary*, *The Gentleman's Diary*, and *Gentleman's Mathematical Companion* come to be listed in preference to such publications as *Comptes rendus de l'Association française pour l'Avancement des Sciences*, the *Reports of the Association for the Improvement of Geometrical Teaching*, *Mathematisk Tidskrift*, and *Tidskrift for Matematik*.

Great Britain has already published at least three lists similar to our union list: (1) Royal Society, Subject Index, Pure Mathematics, which devotes 41 pages to indication of the distribution of 701 serials in 23 libraries; (2) Royal Society, Subject Index, Mechanics, indicating in 58 pages the location of 959 serials in 28 libraries; and (3) W. J. Greenstreet's Catalogue of Current Mathematical Journals (40 pages), compiled for the Mathematical Association and listing about 180 periodicals in 49 libraries.

In this country, apart from printed catalogues of special libraries, the mathematician has had at his disposal till now but two lists: (1) H. C. Bolton's "Catalogue of Scientific and Technical Periodicals 1665-1895"* for 130 libraries in different parts of the country; and (2) a general list for 36 libraries in a limited district, namely, A List of Periodicals, Newspapers, Transactions, and other serial publications currently received in the principal libraries of Boston and the vicinity. Boston, Boston Public Library, 1897. 143 pages.†

While grateful for what we have, it is to be hoped that some one will feel disposed soon to publish for America a union list of all serials indexed by the Royal Society Subject Index, Pure Mathematics, and by the *A* volumes of the International Catalogue of Scientific Literature. Such a list would include four or five times as many titles as the union list under review, but, with a scheme of abbreviations similar to that employed in the Royal Society Subject Index, its size need not be more

* Second edition, Washington, 1897. (Smithsonian Miscellaneous Collections, No. 1076.)

† A new edition of this work including three or four times as many titles, and referring to 72 libraries, is now in course of preparation.

than twice as large. Considering the whole range of mathematics it is somewhat unsafe to claim that any large number of these titles is alone "desirable from the bibliographical and historical standpoint."

A similar list for mechanics, aeronautics, geodesy, ballistics, navigation, and astronomy is also a decided desideratum in these days.

R. C. ARCHIBALD.

July, 1918.

NOTES.

AT the annual meeting of the American Mathematical Society, to be held at Chicago on December 27-28, President DICKSON will deliver his retiring address, on "Mathematics in War Perspective." Abstracts and titles of papers intended for presentation at this meeting should be in the hands of the Acting Secretary, Professor E. J. MOULTON, 909 Colfax Street, Evanston, Ill., by December 2. The meeting will be immediately preceded by that of the Mathematical Association of America, and a joint session will be held on December 27.

THE seventy-first meeting of the American association for the advancement of science will be held at Johns Hopkins University, Baltimore, Md., December 27-31. G. D. BIRKHOFF is vice-president, and F. R. MOULTON secretary of Section A.

THE opening (September) number of volume 20 of the *Annals of Mathematics* contains the following papers: "Functions of limited variation and Lebesgue integrals," by G. P. HORTON; "On the Teixeira construction of the unicursal cubic," by N. ALTSHILLER; "The functional equation $f[f(x)] = g(x)$," by G. A. PFEIFFER; "The existence of the functions of the elliptic cylinder," by MARY F. CURTIS; "The gamma function in the integral calculus," by T. H. GRONWALL.

THE closing (October) number of volume 40 of the *American Journal of Mathematics* contains: "Theta modular groups

determined by point sets," by A. B. COBLE; "On the asymptotic solution of the non-homogeneous linear differential equation of the n th order. A particular solution," by W. V. N. GARRETSON; "A collineation group isomorphic with the group of the double tangents of the plane quartic," by C. C. BRAMBLE; "Proof of Pohlke's theorem and its generalizations by affinity," by ARNOLD EMCH; "Arithmetical theory of certain Hurwitzian continued fractions," by D. N. LEHMER.

THE following 24 doctorates with mathematics as major subject were conferred by American universities in the academic year 1917-1918; the title of the dissertation is added in each case: I. A. BARNETT, Chicago, "Differential equations with a continuous infinitude of variables"; R. F. BORDEN, Illinois, "On the Laplace-Poisson mixed equation"; H. E. BRAY, Rice Institute, "A Green's theorem in terms of Lebesgue's integral"; TERESA COHEN, Johns Hopkins, "An investigation of plane quartic curves"; H. H. DALAKER, Cornell, "On the automorphic functions of the group $(0, 3; 2, 4, 6)$ "; H. D. FRARY, Illinois, "The Green's function of a plane contour"; G. H. HALLETT, Pennsylvania, "Linear order in three-dimensional euclidean and double elliptic space"; ANNA M. HOWE, Cornell, "The classification of plane involutions of order three"; GLENN JAMES, Columbia, "Some theorems on the summation of divergent series"; J. M. KINNEY, Chicago, "The general theory of congruences without any preliminary integrations"; E. P. LANE, Chicago, "Conjugate systems with indeterminate axis of curves"; J. E. McATEE, Chicago, "Modular invariants of a quadratic form for a prime power modulus"; F. R. MORRIS, California, "Classification of involutory cubic space transformations"; W. P. OTT, Chicago, "The general type of the brachistochrone with variable end points"; C. P. PAINE, Wisconsin, "Modes of air motion and the equations of the general circulation of the earth's atmosphere"; O. J. RAMBLER, Catholic University, "Three-cusped. hypocycloids fulfilling certain assigned conditions"; JOSEPHINE R. (Mrs. E. D.) ROE, Syracuse, "Interfunctional expressibility problems of symmetric functions, with tables"; L. J. ROUSE, Michigan, "A contribution to the question of linear dependence in linear integral equations"; H. M. SHOEMAKER, Pennsylvania, "A generalized equation of the vibrating membrane expressed in curvilinear coördinates";

L. S. SHIVELY, Chicago, "A new basis for the metric theory of congruences"; W. G. SIMON, Chicago, "On the solution of certain types of linear differential equations in infinitely many variables"; M. G. SMITH, Illinois, "On the zeros of functions defined by homogeneous linear differential equations containing a parameter": MARY A. SZNYTHER, California, "The hypersurface of the second degree in four-dimensional space"; J. S. TAYLOR, California, "A set of five postulates for Boolean algebras in terms of the operation 'exception.'"

THE Paris academy of sciences announces the award of the following prizes for work in pure and applied mathematics for the year 1918: Poncelet prize (2,000 francs) to JOSEPH LARMOR, of Cambridge University, for the totality of his mathematical researches. Francoeur prize (1,000 francs) to P. MONTEL, of the University of Paris, for his investigation of series of analytic functions. Lalande prize (540 francs) to A. BELOPSLKÝ, of the Pulkowa observatory, for his contributions to the application of spectrum analysis to astronomy. Valz prize (460 francs) to F. SY, of the Algiers observatory, for the totality of his works on astronomy. Janssen prize (medal) to S. CHEVALIER, director of the Shanghai observatory, for his researches in physical astronomy.

AMONG the mathematicians associated with Major Oswald Veblen at the Aberdeen proving ground are A. A. Bennett, Texas; H. F. Blichfeldt, Stanford; G. A. Bliss, Chicago; T. H. Gronwall; C. R. Dines and C. N. Haskins, Dartmouth; Dunham Jackson, Harvard; H. H. Mitchell, Pennsylvania; W. H. Roever, Washington (St. Louis). With Major F. R. Moulton in the ordnance department at Washington are J. W. Alexander, Princeton; Thomas Buck, California; W. D. MacMillan, Chicago; J. F. Ritt and Caroline E. Seely, Columbia; H. L. Smith, Princeton; H. S. Vandiver, Philadelphia. Commissions in the ordnance department of the army have been given to First Lieutenant Thomas Buck, Captain H. H. Mitchell, Captain Dunham Jackson, Major W. D. Macmillan. Dr. J. R. Musselman, Illinois, has been appointed first lieutenant on the statistics branch of the general staff. Dr. D. F. Barrow, Yale, has entered the government service. Dr. J. N. Rice, Catholic University, is serving in the national army. Mr. J. J. Nassau, Syracuse, is in a regiment of engineers in France. Professor

Arnold Dresden, Wisconsin, sailed for France in September for service with the Red Cross.

PROFESSOR W. R. LONGLEY, of Yale University, is on leave of absence and is engaged in the ballistic division of the Dupont Company in Delaware.

MR. G. H. SCOTT has been appointed professor of mathematics in Doane College.

AT the University of Illinois Dr. ELIZABETH B. GRENNAN and Dr. JOSEPHINE GLASGOW have been appointed instructors in mathematics and Mr. L. E. YAEGER assistant in mathematics. Mr. C. H. RICHARDSON, assistant in mathematics, has been appointed professor of mathematics in Georgetown College, Kentucky.

AT Cornell University Mrs. D. NAYLOR has been appointed instructor in mathematics. Instructor E. P. FRALEIGH has resigned to engage in the ordnance work at the Aberdeen proving ground. Seven members of the faculty have been temporarily transferred from other departments to assist in the mathematical instruction.

AT the University of Missouri Mr. F. DUNCAN, Miss Z. FERGUSON, and Mr. A. GROSSMAN have been appointed instructors in mathematics.

DR. F. D. MURNAGHAN, of the Rice Institute, has been appointed associate in applied mathematics at Johns Hopkins University.

AT the University of Pennsylvania, Professor H. B. EVANS, of the department of mathematics, has been made dean of the Towne Scientific School.

PROFESSOR L. LINDSEY, of Syracuse University, has been promoted to an associate professorship of applied mathematics.

DR. E. W. CHITTENDEN, of the University of Illinois, has been appointed assistant professor of mathematics in the University of Iowa.

PROFESSOR T. O. WALTON, of William and Vashti College, has been appointed professor of mathematics in the Colorado School of Mines.

At the University of Rochester assistant professor C. W. WATKEYS has been promoted to a full professorship of mathematics.

PROFESSOR H. R. PHALEN, of Berea College, has been appointed instructor in mathematics in the Armour Institute of Technology.

DR. G. W. SMITH, of Beloit College, has been appointed instructor in mathematics in the University of Kentucky.

DR. W. G. SIMON, of the University of Chicago, has been appointed instructor in mathematics in Adelbert College.

DR. L. L. SILVERMAN, of Cornell University, has been appointed assistant professor of mathematics in Dartmouth College.

PROFESSOR C. A. BARNHART, formerly of Carthage College, has been appointed professor of mathematics in the University of New Mexico.

DR. F. R. MORRIS has been appointed to an instructorship in mathematics in the University of California.

PROFESSOR J. H. GRAF, of the University of Bern, died recently at the age of sixty-six years.

PROFESSOR S. LATTÈS, of the University of Toulouse, died July 5, 1918, at the age of forty-three years.

PROFESSOR C. WOLF, of the University of Paris, died in the summer of 1918 at the age of ninety-one years.

PROFESSOR O. HENRICI, of the City and Guilds Technical College, London, died August 10, at the age of seventy-eight years.

MR. H. W. POWELL, tutor in mathematics in the college of the City of New York, died July 23, 1918. Mr. Powell had been a member of the American Mathematical Society since 1907.

PROFESSOR H. G. KEPPEL, head of the department of mathematics in the University of Florida, died October 5 at the age of fifty-two years. Professor Keppel had been a member of the American Mathematical Society since 1897.

COL. E. W. BASS, professor of mathematics at West Point from 1878 to 1898, died November 6 at the age of seventy-six years.

DR. ARTEMAS MARTIN, of the U. S. Coast Survey, died November 7 at the age of eighty-four years. Dr. Martin had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAMS (O. S.). General theory of the Lambert conformal conic projection. (U. S. Coast and Geodetic Survey, Special Publication No. 53.) Washington, Government Printing Office, 1918. 38 pp. Paper.

\$0.10

BOUTROUX (E.). See GOBLOT (E.).

DUCLOUT (J.). Temas elegidos de matemáticas elementales. La serie de Taylor y la teoría general de las series infinitas, para funciones de una variable real. Desarrollo de las ideas esbozadas por F. Klein en "Matemáticas elementales desde un punto de vista superior," Buenos Aires, Centro Estudiantes de Ingeniería, 1917.

GOBLOT (E.). Traité de logique. Préface de E. Boutroux. Paris, Librairie Armand Colin, 1918. 8vo. 24 + 412 pp. Fr. 9.60

HARNACK (A.). See SERRET (J. A.).

KLEIN (F.). See DUCLOUT (J.).

KOREN (J.). The history of statistics. New York, Macmillan, 1918. 12 + 773 pp. \$7.50

MCCLENON (R. B.). Introduction to the elementary functions by R. B. McClenon with the editorial cooperation of W. J. Rusk. Boston, Ginn, 1918. 8vo. 9 + 244 pp. \$1.80

MORÁVEK (G.). Allgemeine Beweise der Gültigkeit des letzten Fermatschen Satzes. Mit einem Anhang über Pythagoräische Zahlen. Prag, G. Morávek, 1916. 8vo. 18 pp. M. 1.00

- MULDER (P. J.). Kirkman-Systemen. (Diss.). Leyden, P. J. Mulder, 1917. 4to. 15 + 312 pp.
- POST (W. C.). Over de reductie van Abelsche integralen tot elliptische. (Diss.). Leyden, E. J. Brill, 1917. 4to. 10 + 174 pp.
- ROYAL Society Catalogue of Scientific Papers. Fourth series (1884 to 1900), vol. 16: I—Marbut. Cambridge, University Press, 1918. 4to. 6 + 1054 pp. £5 s. 5
- RUSK (W. J.). See McCLENON (R. B.).
- SARKAR (B. K.). Hindu achievements in exact science. A study in the history of scientific development. New York, Longmans, 1918. 8vo. 14 + 82 pp. \$1.00
- SCHEFFERS (G.). See SERRET (J. A.).
- SERRET (J. A.) und Scheffers (G.). Lehrbuch der Differential- und Integralrechnung. Nach A. Harnacks Uebersetzung. 1ter Band: Differentialrechnung. 6te und 7te Auflage. Leipzig, Teubner, 1915. 17 + 670 pp. Geh. M. 13.00
- SLATE (F.). The fundamental equations of dynamics and its main coordinate systems vectorially treated and illustrated from rigid dynamics. Berkeley, University of California Press, 1918. 10 + 233 pp.

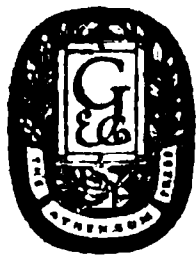
II. ELEMENTARY MATHEMATICS.

- ALZÁA (F. de) y JAIME (F. D.). Elementos de trigonometria rectilinea y esférica. Buenos Aires, A. Garcia Santos, 1918.
- DRESSLER (H.). Elementares Rechnen und bürgerliche Rechnungsarten für die Unterstufe des Seminars. 3te Auflage. Dresden-Blasewitz, Bleyl und Kaemmerer, 1917. Geh. M. 2.85
- FISCHER (P. B.) und ZÜHLKE (P.). Deutschland und der Weltkrieg. Tatsachen und Zahlen aus den Kriegsjahren 1913–1917. Leipzig, Teubner, 1917. 117 pp. Geh. M. 1.60
- HESSENBERG (G.). Ebene und sphärische Trigonometrie. 3te Auflage. Neudruck. (Sammlung Göschen). Berlin, Göschen, 1917. 171 pp.
- JAIME (F. D.). See ALZÁA (F. de).
- LENNES (N. J.). See SUTTON (C. W.).
- LIETZMANN (W.). Aufgabensammlung und Leitfaden für Arithmetik, Algebra und Analysis. Ausgabe A: Für Gymnasien. Unterstufe. Leipzig, Teubner, 1917. 261 pp. Geb. M. 2.80
- SEYFFARTH (W.). Allgemeine Arithmetik und Algebra. 5te Auflage. Blasewitz, Bleyl und Kämmerer, 1917. 174 pp. Geh. M. 2.20
- SUTTON (C. W.) and LENNES (N. J.). Business arithmetic. Boston, Allyn and Bacon, 1918. 8vo. 6 + 466 + 8 pp. Cloth.
- ZÜHLKE (P.). See FISCHER (P. B.).

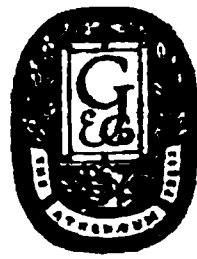
III. APPLIED MATHEMATICS.

- ADAMS (O. S.). Lambert projection tables for the United States. Washington, Government Printing Office, 1918. 8vo. 243 pp. Paper. \$0.25
- ALLINGHAM (W.). See LECKY (S. T. S.).

- ANDERSON (M. H.). Elements of pilotage and navigation. 2d edition, revised. Portsmouth, Gieves, 1917. 5s.
- BOCK (H.). Die Uhr, Grundlagen und Technik der Zeitmessung. 2te Auflage. Leipzig, Teubner, 1917. 121 pp. 7s. 6d.
- BUCK (P. C.). Acoustics for musicians. Oxford, Clarendon Press, 1918. 152 pp. 7s. 6d.
- CARD (S. F.). Navigation notes and examples. 2d edition. London, Arnold, 1917. 10s. 6d.
- CUGLE (C. H.). Simple rules and problems in navigation. 2d edition. New Orleans, Marine Publishing Company, 1918. 290 pp. and 4 plates. \$3.50
- DEETZ (C. H.). The Lambert conformal conic projection with two standard parallels, including a comparison with the Bonne and polyconic projections. Washington, Government Printing Office, 1918. Royal 8vo. 61 pp. and 7 plates. Paper. \$0.75
- Lambert projection tables with conversion tables supplement to The Lambert conformal conic projection with two standard parallels. Washington, Government Printing Office, 1918. Royal 8vo. 84 pp. Paper. \$0.15
- GRANT (H. C. J.). Pocketbook of practical navigation. Portsmouth, Gieves Publishing Company, 1917. 7s. 6d.
- LECKY (S. T. S.). Practical wrinkles in navigation. 18th edition revised by W. Allingham. Liverpool, Philip, 1917. 25s.
- McKREADY (K.). Die Sternwelt (Meister der Naturwissenschaften, N. 11). Leipzig, Barth, 1917. M. 0.45
- NEUENDORFF (R.). Praktische Mathematik. 1ter Teil, 2te Auflage. (Aus Natur und Geisteswelt, Nr. 341). Leipzig, Teubner, 1918. Geb. M. 1.50
- NICHOLLS (—.). Concise guide to Board of Trade examinations for masters and mates. 14th edition. Glasgow, J. Brown, 1917. 10s. 6d.
- ROTH (A.). Grundlagen der Elektrotechnik. (Aus Natur und Geisteswelt, Nr. 391). 2te Auflage. Leipzig, Teubner, 1917. 143 pp.
- SCHUDEISKY (A.). Leitfaden für den neuzeitlichen Linearzeichenunterricht. Für die Hand des Schülers. Leipzig, Teubner, 1917. 39 pp. Geh. M. 1.40
- SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik 1ter Teil, für die mittleren Klassen höherer Schulen. 3te Auflage. Leipzig, Teubner, 1917. 8vo. 204 pp. Geb. M. 2.60
- TAITS (—.). New guide to Board of Trade examinations in navigation and nautical astronomy. Glasgow, Munro, 1917. 8s.



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**BULLETIN OF THE AMERICAN
MATHEMATICAL SOCIETY**

J. M. WILLARD

STATE COLLEGE, PA.

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two-hundredth regular meeting of the Society was held in New York City on Saturday, October 26, 1918. War conditions and the prevailing epidemic cooperated to reduce the attendance, which included the following eleven members:

Professor E. W. Brown, Professor F. N. Cole, Professor T. S. Fiske, Mr. T. C. Fry, Professor O. E. Glenn, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor P. H. Linehan, Professor R. L. Moore, Professor H. W. Reddick.

Professor E. W. Brown presided at the morning session, and Professor O. E. Glenn at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor R. A. Arms, Juniata College; Professor M. D. Earle, Furman University; Professor Ernest Flammer, Queen's University; Professor Gillie A. Larew, Randolph-Macon Woman's College; Dr. Flora E. Le Sturgeon, Mt. Holyoke College; Professor John Matheson, Queen's University; Dr. F. R. Morris, University of California; Professor Susan M. Rambo, Smith College; Dr. W. G. Simon, Adelbert College. One application for membership in the Society was received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election.

A committee was also appointed to collect subscriptions from members of the Society and others for the purpose of establishing at Harvard University a suitable memorial of the late Professor Maxime Bôcher, President of the Society in 1909-1910.

The Chicago meeting in the Christmas holidays was designated by the Council as the annual meeting of the Society for this year, the usual eastern meeting being cancelled. Members attending the Baltimore meeting of the American Association were invited to read papers before Section A, having previously registered titles and abstracts with the Secretary for record in the report of the annual meeting.

The Southwestern Section decided not to hold its usual meeting this year. The February, 1919, meeting of the Society will also be omitted. The usual meetings will be held in April.

The following papers were read at the October meeting:

(1) Dr. J. E. McATEE: "The transformation of a regular group into its conjoint."

(2) Professor E. W. CHITTENDEN: "On the Heine-Borel property in the theory of abstract sets."

(3) Professor GILLIE A. LAREW: "Necessary conditions for the problem of Mayer in the calculus of variations."

(4) Professor D. M. Y. SOMERVILLE: "Quadratic systems of circles in non-euclidean geometry."

(5) Professor M. B. PORTER: "Derivativeless continuous functions."

(6) Dr. G. H. HALLETT, JR.: "Concerning the definition of a simple continuous arc."

(7) Professor R. L. MOORE: "A characterization of Jordan regions by properties having no reference to their boundaries."

(8) Professor R. L. MOORE: "Concerning simple continuous curves."

(9) Professor EDWARD KASNER: "Fields of force and Monge equations."

Dr. Hallett's paper was communicated to the Society through Professor R. L. Moore. In the absence of the authors the papers of Dr. McAtee, Professor Chittenden, Professor Larew, Professor Somerville, Professor Porter, and Dr. Hallett were read by title. Abstracts of the papers follow below.

1. The statement that there is a transformation of order 2 that transforms a regular substitution group into its conjoint occurs in the literature. In this paper Dr. McAtee exhibits such a transformation.

2. Professor Chittenden's paper appeared in full in the November BULLETIN.

3. Since A. Mayer formulated that problem of the calculus of variations usually called by his name, considerable attention has been given to the deduction of the Euler-Lagrange equations of the problem as a first necessary condition. Little attention, subsequent to the original paper of Mayer, has been

paid to the necessary conditions analogous to those of Legendre and Jacobi or to the deduction of a necessary condition analogous to that of Weierstrass. It is the purpose of Professor Larew's paper to inquire systematically into the question of necessary conditions. In the investigation of the corner-point condition for the so-called "discontinuous solutions," the theorem on the necessary condition of Euler is extended to include arcs which are continuous, but which may have a finite number of corners. A formulation and proof is supplied for the necessary condition of Weierstrass, and the Legendre condition follows directly. The Jacobi condition is deduced in much more simple fashion than usual by an application of the Euler equations and the corner-point condition to the second variation.

4. Professor Somerville's paper discusses the quadratic systems of circles with one or two parameters in non-euclidean geometry. The general one-parameter system consists of one of three systems of circles all having double contact with a fixed conic K . The limiting points are a pair of foci, and the limiting lines are a pair of focal lines of K . Coaxal and homocentric systems are degenerate cases, and a third degenerate case consists of a system of equal circles. The two-parameter quadratic system is found to be composed of one-parameter systems whose conic envelopes all have double contact with a fixed conic. A specialized system is noteworthy in which every circle cuts a fixed circle orthogonally; another system, reciprocal to this, has the property that every circle is tangentially distant a quadrant from a fixed circle.

5. Professor Porter's paper deals with continuous functions which either have nowhere a derivative or only possess derivatives inside an interval or at a point. A simple method is developed for constructing such functions in a large variety and it is shown that in the case of Weierstrass's type the condition that a be an odd integer is unnecessary.

6. In a paper entitled "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, volume 33 (1911), pages 285-326, N. J. Lennes gives the following definition of a simple continuous arc. "A continuous simple arc connecting two points

A and B , $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing A and B such that no connected proper subset of $[A]$ contains A and B ." The purpose of Dr. Hallett's paper is to show that the word "bounded" in this definition is superfluous.

7. Schoenflies* has formulated a set of conditions under which the common boundary of two domains will be a simple closed curve. A different set has been given by J. R. Kline (cf. this BULLETIN, July, 1918, page 471). Carathéodory† has secured conditions under which the boundary of a single domain will be such a curve. In each of these treatments conditions are imposed (1) on the boundary itself, (2) regarding the relation of the boundary to the domain or domains in question. In the present paper Professor Moore establishes the following theorem in which all the conditions imposed are on the domain R alone:

In order that a simply connected limited two-dimensional domain R should have a simple closed curve as its boundary it is necessary and sufficient that R should be uniformly connected im kleinen.

A point set M is said to be uniformly connected im kleinen‡ if for every positive number ϵ there exists a positive number δ_ϵ such that if P_1 and P_2 are two points of M at a distance apart less than δ_ϵ they lie together in a connected subset of M every point of which is at a distance less than ϵ from P_1 .

8. Various definitions have been given of simple continuous arcs, simple closed curves and open continuous curves. In the present paper Professor Moore defines these terms from a point of view which, as far as he knows, is different from any hitherto employed for this purpose. He makes use of the notion of the boundary of a point set with respect to a point set that contains it. If M is a proper subset of N , a boundary point of M with respect to N is a point which is a point or a limit point of M and also a point or a limit point of $N - M$.

Definition 1. A simple continuous closed curve is a continuous§ point set M every continuous proper subset of which has two and only two boundary points with respect to M .

* *Göttinger Nachrichten*, 1902, p. 185.

† *Ma'h. Annalen*, vol. 73 (1912-13), p. 366.

‡ Cf. H. Hahn, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318-322.

§ A set of points is said to be continuous if it is closed and connected.

Definition 2. A simple continuous open curve is a continuous point set M every continuous bounded subset of which has just two boundary points with respect to M .

Definition 3. If A and B are two distinct points, a simple continuous arc from A to B is a continuous point set M containing A and B such that (1) no continuous subset of M has more than two boundary points with respect to M , (2) a subset K of M has just two boundary points with respect to M if and only if K contains neither A nor B .

He shows that these point sets may also be defined in the following simple manner.

Definition 1'. A simple continuous closed curve is a continuous bounded point set which is disconnected by the omission of any two of its points.

Definition 2'. A simple continuous open curve is a continuous point set which is disconnected by the omission of any one of its points.

Definition 3'. If A and B are two distinct points, a simple continuous arc from A to B is a continuous bounded point set which contains A and B and is disconnected by the omission of any one of its points other than A or B .

Every simple closed curve as defined in 1 or 1' is in one-to-one continuous correspondence with a circle. Every simple continuous open curve as defined in 2 or 2' is in one-to-one continuous correspondence with a straight line and every simple continuous arc as defined in 3 or 3' is in one-to-one continuous correspondence with a linear interval. As has been shown by Dr. G. H. Hallett, Jr., in Lennes's definition of a simple continuous arc the condition that the point set in question should be *bounded* is superfluous. In Definition 3' this is not the case.

9. After observing that the ∞^5 trajectories defined by a general field of force in space cannot all satisfy a Monge equation, Professor Kasner raises the question under what conditions this is possible for ∞^4 of the trajectories. It is shown that the Monge equation must be linear (that is, Pfaffian) and that it must define a null system. The direction of the force vector at any point must be in the null system. All forces of this type are determined explicitly.

F. N. COLE,
Secretary.

ON THE EVALUATION OF THE ELLIPTIC TRANSCENDENTS η_2 AND η_2' .

BY PROFESSOR HARRIS HANCOCK.

With Halphen* write

$$f(s) = 4s^3 - g_2s - g_3 = 4(s - e_1)(s - e_2)(s - e_3) = S,$$

where e_2 is a real quantity and e_1, e_3 are conjugate imaginaries such that $e_1 - e_3$ is a positive pure imaginary. The discriminant $\Delta = g_2^3 - 27g_3^2$ is here negative. The elliptic differential equation in this notation is $ds^2/du^2 = 4s^3 - g_2s - g_3$, whose integral is

$$u = \int_s^\infty \frac{ds}{\sqrt{f(s)}}.$$

If the lower limit of this integral be considered as a function of the integral, that is of u , we may write $s = \wp u$.

As is well known, the quantity ω_2 is defined by

$$(1) \quad \omega_2 = \int_{e_2}^\infty \frac{ds}{\sqrt{f(s)}}; \quad \wp \omega_2 = e_2, \quad \wp' \omega_2 = 0.$$

If further we write

$$f_1(s) = 4s^3 - g_2s + g_3 = 4(s + e_1)(s + e_2)(s + e_3),$$

the quantity ω_2' is defined through the relations

$$(2) \quad \frac{\omega_2'}{i} = \int_{-e_2}^\infty \frac{ds}{\sqrt{f_1(s)}}; \quad \wp \omega_2' = e_2, \quad \wp' \omega_2' = 0.$$

Note that ω_2' is a pure imaginary.

It may be shown that the three half-periods

$$\omega_1 = \frac{1}{2}(\omega_2 - \omega_2'), \quad \omega_2, \quad \omega_3 = \frac{1}{2}(\omega_2 + \omega_2'),$$

correspond to the quantities e_1, e_2, e_3 above, the relations being the same as are those for $\omega, \omega'' = \omega + \omega', \omega'$ in the usual notation where all three of the roots e_1, e_2, e_3 are real.

Write

$$\frac{d}{du} \zeta u = - \wp u,$$

* Fonctions Elliptiques, vol. 1, chap. 6.

note that $\zeta(-u) = -\zeta u$ and that the quantities η_1 and η_3 are defined through the relations $\zeta\omega_1 = \eta_1$ and $\zeta\omega_3 = \eta_3$. Further write

$$\eta_1 = \frac{1}{2}(\eta_2 - \eta_2'), \quad \eta_3 = \frac{1}{2}(\eta_2 + \eta_2').$$

In practically all the text books on this subject it is shown that the two half-periods ω_1 and ω_2 play with respect to the function ζu the same rôle; when the discriminant is negative, as is played by the half-periods ω and ω' when the discriminant is positive; and the constants η_1 and η_3 which correspond to these half-periods are connected with η_2 and η_2' by the same relations which connect ω_1, ω_3 with ω_2, ω_2' . The values of η_2 and η_2' may be derived as follows:

In my monograph on Elliptic Integrals, page 51, formulas (39) and (40), it is shown that

$$(3) \quad \int_a^x \frac{dx}{\sqrt{(x-\alpha)[(x-\rho)^2 + \sigma^2]}} = \frac{1}{\sqrt{M}} \operatorname{cn}^{-1} \left[\frac{M - (x-\alpha)}{M + (x-\alpha)}, k \right],$$

where

$$k^2 = \frac{1}{2} - \frac{1}{2} \frac{\alpha - \rho}{M}, \quad M^2 = (\rho - \alpha)^2 + \sigma^2;$$

and

$$(4) \quad \int_x^\alpha \frac{dx}{\sqrt{(\alpha-x)[(x-\rho)^2 + \sigma^2]}} = \frac{1}{\sqrt{M}} \operatorname{cn}^{-1} \left[\frac{M - (\alpha-x)}{M + (\alpha-x)}, k' \right],$$

where

$$k'^2 = \frac{1}{2} + \frac{1}{2} \frac{\alpha - \rho}{M}; \quad k^2 + k'^2 = 1, \quad 2kk' = \frac{\sigma}{M}.$$

These two formulas may be made the foundation of the whole so-called Weierstrassian theory, with negative discriminant.

To save the reader the trouble of deriving them and simply as a verification, write

$$u = \operatorname{cn}^{-1} \left[\frac{M - (x-\alpha)}{M + (x-\alpha)} \right]$$

or

$$\operatorname{cn} u = \frac{M - (x - \alpha)}{M + (x - \alpha)} = \cos \varphi, \text{ where } \varphi \text{ is put} = \operatorname{am} u.$$

It follows at once that

$$x - \alpha = M \frac{1 - \cos \varphi}{1 + \cos \varphi}, \quad dx = 2M \frac{\sin \varphi d\varphi}{(1 + \cos \varphi)^2},$$

$$(x - \rho)^2 + \sigma^2 = \frac{2M^2(1 + \cos^2 \varphi) + 2(\alpha - \rho)M \sin^2 \varphi}{(1 + \cos \varphi)^2}.$$

These values substituted in (3) cause that integral to become

$$\frac{1}{\sqrt{M}} \int_0^{\varphi} \frac{dx}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad \text{where } k^2 = \frac{1}{2} - \frac{1}{2} \frac{\alpha - \rho}{M}.$$

The indicated transformation reduces the integral (4) to the corresponding Legendre normal form.

Note that when $x = \alpha$, then $\varphi = \frac{1}{2}\pi$. From (3) and (4) it follows at once that

$$(5) \quad \omega_2 = \frac{K}{\sqrt{M}}, \quad \omega_2' = \frac{iK'}{\sqrt{M}},$$

where

$$K = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad K' = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k'^2 \sin^2 \varphi}},$$

k^2 and k'^2 having the values already indicated.

If one wished to evaluate an elliptic integral of the second kind in which there appeared a denominator, as in the integrals (3) and (4), similar reductions would transform it into the corresponding Legendre normal form and its value could be taken at once from the Legendre tables.

If however the Weierstrassian formulas are introduced, we may proceed as follows: From the definition above

$$\zeta u = - \int \wp u du;$$

or, since $\wp u = s$ and

$$du = \frac{ds}{\sqrt{4s^3 - g_2s - g_3}},$$

we may write*

$$\zeta u = \int \frac{s ds}{\sqrt{4s^3 - g_2 s - g_3}},$$

with suitable choice of the constant in the lower limit.

If we put $e_1 = \rho + i\sigma$, $e_3 = \rho - i\sigma$, it is seen that

$$\begin{aligned} \int_{e_1}^{e_3} \frac{s ds}{\sqrt{S}} &= \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{s ds}{2 \sqrt{(s - e_2)(s - e_1)(s - e_3)}} \\ &= \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{s ds}{2 \sqrt{(s - e_2)[(s - \rho)^2 + \sigma^2]}}. \end{aligned}$$

Observe that the left-hand integral is

$$\zeta \omega_3 - \zeta \omega_1 = \frac{1}{2}(\eta_2 + \eta_2') - \frac{1}{2}(\eta_2 - \eta_2') = \eta_2'.$$

In the right-hand integral put $s = \rho - it$ and it follows that

$$(6) \quad -\eta_2' = \int_{t=\sigma}^{t=-\sigma} \frac{(\rho - it)idt}{2 \sqrt{(3\rho - it)(t + \sigma)(\sigma - t)}}.$$

To evaluate this integral write with Lucien Lévy† (see page 95)

$$\frac{1}{\sqrt{3\rho - it}} = \alpha + i\beta.$$

We have at once

$$\frac{3\rho + it}{9\rho^2 + t^2} = \alpha^2 - \beta^2 + i2\alpha\beta,$$

or

$$\alpha^2 - \beta^2 = \frac{3\rho}{9\rho^2 + t^2}, \quad 2\alpha\beta = \frac{t}{9\rho^2 + t^2},$$

$$\alpha^2 + \beta^2 = \frac{1}{9\rho^2 + t^2}.$$

As the sign of α is arbitrary, we shall take it positive; while the sign of β , from the second of the formulas just written, is the same as the sign of t . This may be indicated by the notation ϵ_t .

* See for example Schwarz, Formeln, etc., p. 87.

† Précis élémentaire de la Théorie des Fonctions Elliptiques, published by Gauthier-Villars, Paris, 1898.

It follows at once that

$$\alpha = \frac{1}{\sqrt{2}} \frac{\sqrt{3\rho + \sqrt{9\rho^2 + t^2}}}{\sqrt{9\rho^2 + t^2}},$$

$$\beta = \epsilon_t \frac{\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}}{\sqrt{9\rho^2 + t^2}}.$$

Write $(\rho - it)i(\alpha + i\beta)$ in the form $\alpha t - \beta\rho + i(\alpha\rho + \beta t)$ and put $\eta_2' = A + iB$. From formula (6) it is seen that

$$-A = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2} \sqrt{\sigma^2 - t^2}}$$

$$\times [t\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} - \rho\epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}],$$

$$-B = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2} \sqrt{\sigma^2 - t^2}}$$

$$\times [\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + \epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}].$$

If we write A in the form

$$\int_{-\sigma}^0 + \int_0^{\sigma}$$

and in either of these integrals make the substitutions $t = -\tau$, it is seen that the one is the negative of the other, so that $A = 0$. This was evident à priori, since $\eta_2' = \zeta(\omega_2')$ is a pure imaginary, ζ being an odd function and ω_2' a pure imaginary.

It also follows that

$$B = \frac{\eta_2'}{i} = - \int_{t=0}^{t=\sigma} \frac{dt}{\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2} \sqrt{\sigma^2 - t^2}}$$

$$\times [\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + t\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}].$$

In this expression put $9\rho^2 + t^2 = x^2$, and it becomes, if $t > 0$,

$$(7) \quad \frac{\eta_2'}{i} = - \int_{x=3\rho}^{x=M} \frac{(x - 2\rho)dx}{\sqrt{2}(x - 3\rho)(M^2 - x^2)},$$

where

$$M^2 = 9\rho^2 + \sigma^2.$$

To evaluate the integral (7), write $x = -t + \rho$, and we have

$$(8) \quad \frac{-\eta_2'}{i} = \sqrt{2} \int_{t=-2\rho}^{t=\rho-M} \frac{(t + \rho)dt}{2\sqrt{(t-t_1)(t-t_2)(t-t_3)}},$$

where

$$t_1 = \rho + M, \quad t_2 = -2\rho, \quad t_3 = \rho - M; \quad t_1 + t_2 + t_3 = 0.$$

Write

$$u = \int_t^\infty \frac{dt}{\sqrt{4(t-t_1)(t-t_2)(t-t_3)}},$$

so that $t = \wp u$, an elliptic function with half-periods, say, $\omega, \omega', \omega'' = \omega + \omega'$.

The above transformations, although somewhat complicated, are instructive and are due to Lévy, loc. cit., pages 96 and 97. In the next substitution, however, he makes a rather curious mistake: For du he writes the value just given and associated with it he writes $t = \wp(u + \omega'')$. This of course vitiates his results, which are repeated under the heading "Principal Formulas," page 221. The formula (8) becomes*

$$\begin{aligned} \frac{-\eta_2'}{i} &= \sqrt{2} \int_{\omega''}^{\omega'} (\wp u + \rho) du \\ &= \sqrt{2}[-\zeta\omega' + \zeta\omega'' + \rho\omega' - \rho\omega''] = \sqrt{2}(\eta - \rho\omega). \end{aligned}$$

The values of η and η' expressed in the Legendre normal forms are given in most text-books; see for example Schwarz, loc. cit., page 34. Schwarz fails, however, to give the corresponding values of η_2 and η_2' .

It is seen that

$$\begin{aligned} \frac{-\eta_2'}{i} &= \sqrt{2} \left[\sqrt{t_1 - t_3} \left(E' - \frac{t_1 K'}{t_1 - t_3} \right) - \rho \frac{K'}{\sqrt{t_1 - t_3}} \right] \\ &= \sqrt{M} \left[2E' - \frac{M + 2\rho}{M} K' \right]. \end{aligned}$$

* Here again we must be careful regarding the sign of $dt/du = +\sqrt{\quad}$. See, for example, Appell et Lacour, *Fonctions Elliptiques*, p. 72.

The integral in formula (8) may be again calculated as follows: Write*

$$\sin^2 \varphi = \frac{M - x}{M - 3\rho},$$

so that

$$M - x = (M - 3\rho)\sin^2 \varphi,$$

$$x - 3\rho = (M - 3\rho) \cos^2 \varphi, \quad M + x = 2M[1 - k'^2 \sin^2 \varphi],$$

where

$$k'^2 = \frac{1}{2} - \frac{3\rho}{2M}, \quad dx = (3\rho - M)2 \sin \varphi \cos \varphi d\varphi.$$

The integral in question becomes

$$\begin{aligned} & \frac{1}{\sqrt{M}} \int_0^{\pi/2} \frac{(3\rho - M)\sin^2 \varphi + M - 2\rho}{\sqrt{1 - k'^2 \sin^2 \varphi}} d\varphi \\ &= \frac{2M}{\sqrt{M}} \int_0^{\pi/2} \frac{1 - k'^2 \sin^2 \varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}} \\ & \quad - \frac{2M}{\sqrt{M}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}} + \frac{M - 2\rho}{\sqrt{M}} K' \\ &= 2\sqrt{M}E' - \frac{M + 2\rho}{\sqrt{M}} K', \text{ as above.} \end{aligned}$$

Note that $e_2 = -2\rho$. It is seen at once from formulas (1) and (2) above that η_2' is had from η_2 by changing the sign of e_2 and multiplying the value of η_2 by $-i$ and putting k' for k . We thus have

$$\eta_2 = \sqrt{M} \left[2E - \left(1 - \frac{2\rho}{M} \right) K \right], \quad 2\rho = -e_2, \quad M^2 = 9\rho^2 + \sigma^2,$$

$$\eta_2' = -i \sqrt{M} \left[2E' - \left(1 + \frac{2\rho}{M} \right) K' \right],$$

$$\eta_2 \omega_2' - \eta_2' \omega_2 = i[2EK' + 2E'K - 2KK] = i2\pi.$$

The relation to the right is, of course, Legendre's celebrated formula.

* See formula (17), page 45, of my monograph mentioned above.

One can judge from the present paper the little value of the Weierstrassian formulas when the applications of the general theory are involved or whenever any kind of numerical computation is desired. One sees also how easy it is to introduce errors in the calculation. The book of Lévy already mentioned is in most respects an excellent work, certainly from the standpoint of applied mathematics, much of the material being new; but when a substitution involving an elliptic function in the Weierstrassian form is introduced, the book is not free from errors. For example, not to mention many inaccuracies, besides the mistake already cited, it is seen that on page 82 of Lévy's book $e_1 + e_2 + e_3 \neq 0$. The same error is found on page 156, while in the calculation of ζu , the functions introduced on page 104 are incorrectly given. At the end of my larger work, volume 1, the Weierstrassian functions are put in juxtaposition with those of the older theory and it is seen that thereby nothing new is added.

UNIVERSITY OF CINCINNATI,
June 14, 1918.

ON PLANE ALGEBRAIC CURVES WITH A GIVEN SYSTEM OF FOCI.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, April 27, 1918.)

1. Let

$$(1) \quad \phi(u, v, w) = 0$$

be a curve of class n , and

$$(2) \quad u\xi + v\eta + w\zeta = 0$$

a line with the coordinates u, v, w , and $x' = \xi/\zeta, y' = \eta/\zeta$ current cartesian coordinates. Then

$$(3) \quad -1 \cdot \xi - i \cdot \eta + (x + iy)\zeta = 0$$

is a line which passes through the point (x, y) and the circular point I with the slope i . The coordinates of (3) are $u = -1$,

$v = -i, w = x + iy$. Consequently, if these satisfy (1), then the line (3) will be a tangent of (1); and as it passes through the point (x, y) , this point will be a focus of the curve. Hence, to find the real foci of (1), we have to determine real values of x and y in such a manner that

$$(4) \quad \phi(-1, -i, x + iy) = 0.$$

In other words, we have to solve the equation of the n th degree in w ,

$$(5) \quad \phi(-1, -i, w) = 0.$$

The n complex roots of this equation interpreted in the cartesian plane are the required foci.* These are also obtained by writing (4) in the form $f(x, y) + ig(x, y) = 0$, where f and g are real polynomials in x and y , and by finding the intersections of f and g . In this manner we get n^2 solutions of which n are real and correspond to the n roots of (5). The remaining $n(n-1)$ imaginary solutions are the *associates* of the n roots of (5).

2. Conversely, when the foci $z_1, z_2, \dots, z_n$ of a curve are given in a complex plane, we can easily form the equation

$$(6) \quad \prod_{k=1}^n (z - z_k) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n = 0,$$

with these as roots. The curve of the n th class with these as foci has the form

$$(7) \quad \begin{aligned} \phi \equiv & w^n + (\alpha_{11}v + \alpha_{12}u)w^{n-1} + (\alpha_{21}v^2 + \alpha_{22}vu + \alpha_{23}u^2)w^{n-2} \\ & + (\alpha_{31}v^3 + \alpha_{32}v^2u + \alpha_{33}vu^2 + \alpha_{34}u^3)w^{n-3} + \dots \\ & + \alpha_{n1}v^n + \alpha_{n2}v^{n-1}u + \alpha_{n3}v^{n-2}u^2 + \dots + \alpha_{nn}u^n = 0. \end{aligned}$$

The foci are conversely obtained by putting in this equation $u = -1, v = -i$, and solving for w . In this manner from (7) we get

$$(8) \quad \begin{aligned} & w^n + (i\alpha_{11} - \alpha_{12})w^{n-1} + (-\alpha_{21} + i\alpha_{22} + \alpha_{23})w^{n-2} \\ & + (i\alpha_{31} + \alpha_{32} - i\alpha_{33} - \alpha_{34})w^{n-3} + \dots \\ & + (-i)^n\alpha_{n1} - (-i)^{n-1}\alpha_{n2} + (-i)^{n-2}\alpha_{n3} + \dots \\ & + (-1)^n\alpha_{nn} = 0. \end{aligned}$$

* Siebeck: "Ueber eine neue analytische Behandlungsweise der Brennpunkte," *Crelle's Journal*, vol. 64, pp. 175-182 (1865).

In this equation we can determine real values of the α_{jk} ($j, k = 1, 2, 3, \dots, n$) in an infinite number of ways in such a manner that (8) becomes identical with (6). For any such a set of α_{jk} 's (7) represents a curve of the n th class with the given points as foci.

To obtain the coordinates of (7) in projective point coordinates ξ, η, ζ , we must eliminate w, v, u between (7) and the three equations

$$(9) \quad \xi = \frac{\partial \phi}{\partial u}, \quad \eta = \frac{\partial \phi}{\partial v}, \quad \zeta = \frac{\partial \phi}{\partial w},$$

as is well known.

3. When an n -ic $F(x, y) = 0$ is given, the foci may be found by making the substitution $y = ix + c$, and by establishing the discriminant for the equation $F(x, ix + c) = 0$ in x . This leads to an equation in c whose roots determine the foci. If $c_k = A_k + iB_k$ is such a root, then, as $c = y - ix$, the coordinates of the corresponding focus are $x = -B_k, y = A_k$.

Another method for finding the foci of an n -ic is suggested by the foregoing results: Establish the equation $\phi(u, v, w) = 0$ corresponding to $F(x, y) = 0$ by the well known method of elimination, as suggested above for the converse case.

4. As an example consider the circular cubic*

$$(10) \quad F \equiv (x^2 + y^2)x + yz^2 = 0,$$

whose cartesian equation is obtained by putting $z = 1$. The equation of the equivalent curve of class (6), in line coordinates u, v, w is obtained by eliminating x, y, z between (10) and

$$(11) \quad u = \partial F / \partial x = 3x^2 + y^2, \quad v = 2xy + z^2, \quad w = 2yz.$$

After this rather tedious process the required equation becomes

$$(12) \quad 4w^6 + 24uvw^4 - (u^4 - 30u^2v^2 - 27v^4)w^2 \\ - 4u^3v(u^2 + v^2) = 0.$$

Setting $u = -1, v = -i$, this reduces to

$$(13) \quad w^6 + 6iw^4 - w^2 = 0,$$

* See *Annals of Mathematics*, vol. 14, pp. 57-71 (1912).

whose roots are $w_1, 2, 3, 4 = \pm \frac{1}{2} \sqrt{2}(1 + i)(-3 \pm \sqrt{8})$, $w_5 = 0, w_6 = 0$. The coordinates of the foci F_i of the circular cubic (10) are therefore

$$\begin{aligned} F_1(x_1 = y_1 = 2 - \frac{3}{2} \sqrt{2}), & \quad F_2(x_2 = y_2 = -2 + \frac{3}{2} \sqrt{2}), \\ F_3(x_3 = y_3 = -2 - \frac{3}{2} \sqrt{2}), & \quad F_4(x_4 = y_4 = 2 + \frac{3}{2} \sqrt{2}), \\ F_5(x_5 = y_5 = 0), & \quad F_6(x_6 = y_6 = 0). \end{aligned}$$

F_5 and F_6 coincide with the origin, and form a quadruple focus which lies on the curve. It is called quadruple, since it may be considered as the intersection of two two-fold tangents at the isotropic points.

The foci F_1, F_2, F_3, F_4 of a circular cubic lie on a circle, which in this case degenerates into a straight line, the bisecting line of the first and third quadrant.

5. *Confocal conics*.—Let $F_1(1, 0)$ and $F_2(-1, 0)$ be the foci of a system of confocal conics. Then the determining equation is $w^2 - 1 = 0$, and the system of curves of the second class with these foci is

$$w^2 - \alpha u^2 - (\alpha - 1)v^2 = 0.$$

The corresponding cartesian equation is

$$\frac{x^2}{\alpha} + \frac{y^2}{\alpha - 1} = 1,$$

which, for a variable α , represents a system of confocal conics.

6. *A certain class of curves, admitting the n th roots of unity as foci*. The equation determining these foci is

$$(14) \quad w^n - 1 = 0,$$

and the corresponding curve of class n has the form

$$(15) \quad w^n - (\alpha v^n + \alpha_1 v^{n-1}u + \alpha_2 v^{n-2}u^2 + \alpha_3 v^{n-3}u^3 + \dots + \alpha_n u^n) = 0,$$

in which, when n is even, say $n = 2\lambda$,

$$\alpha(-1)^\lambda + \alpha_2(-1)^{\lambda-1} + \alpha_4(-1)^{\lambda-2} + \dots = 1.$$

For an even λ , this becomes

$$\alpha - \alpha_2 + \alpha_4 - \dots = 1.$$

For any λ , there is $\alpha_1 - \alpha_3 + \alpha_5 - \dots = 0$. When λ is odd, then $-\alpha + \alpha_2 - \alpha_4 + \dots = 1$. When n is odd, say $n = 2\lambda + 1$, then $\alpha - \alpha_2 + \alpha_4 - \dots = 0$, and $\alpha_1 - \alpha_3 + \alpha_5 - \dots = 1$, when λ is odd; $-\alpha_1 + \alpha_3 - \alpha_5 + \dots = 1$, when λ is even.

We shall, in particular, consider the case where (15) has the form

$$(16) \quad w^n - v^{2k} \cdot u^{n-2k} = 0,$$

in which n and k must both be either even or odd in order that (16) may reduce to (14) for $u = -1, v = -i$.

After a rather complicated process of elimination the cartesian equation of this special class of curves with the n th roots of unity as foci becomes

$$(17) \quad x^{n-2k} \cdot y^{2k} = (-1)^{n-2k} \cdot \frac{(2k)^{2k}}{n^n \cdot (n-2k)^{2k-n}},$$

which is an n -ic. For $n = 3, k = 1$, we get the cubic hyperbola $xy^2 = -4/27$. The condition for a proper n -ic in (17) is evidently $n - 2k \geq 1, n \geq 3$.

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QUADRATIC SYSTEMS OF CIRCLES IN NON-EUCLIDEAN GEOMETRY.

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§ 1. The equation of any circle can be written

$$(1) \quad kS - \alpha^2 = 0,$$

where $S = 0$ is the equation of the absolute, and

$$\alpha \equiv lx + my + nz = 0$$

is the equation of the axis, the center being the absolute polar of this line.

If the circle belongs to a one-parameter system, the coefficients k, l, m, n will be functions of the single parameter λ . The degree of the system is the number of circles which pass through a given point, and its class is the number of circles which touch a given straight line.

If

$$S \equiv a_0x^2 + \dots + 2f_0yz + \dots \equiv (S_0x)^2 = 0,$$

$$\Sigma \equiv A_0\xi^2 + \dots + 2F_0\eta\zeta + \dots \equiv (\Sigma_0\xi)^2 = 0$$

represent the point and line equations of the absolute, the line equation of the circle (1) is

$$(2) \quad k\Sigma - (b_0n^2 + c_0m^2 - 2f_0mn)\xi^2 - \dots \\ + 2(f_0l^2 + a_0mn - h_0nl - g_0lm)\eta\zeta + \dots = 0,$$

which may also be written

$$\{(\Sigma_0l)^2 - k\Delta_0\}\Sigma - \{\Sigma(A_0l + Hm + Gn)\xi\}^2 = 0.$$

The system of circles will be of the first degree only if l, m, n are constant, and $k = (a\lambda + b)/(c\lambda + d)$; it is then also of the first class. Hence *the only linear system of circles is a concentric system.*

§ 2. If l, m, n are of degree r in λ , while k is of degree $2r$ or $2r + 1$, the system is of degree $2r$ or $2r + 1$ respectively, and its class is also $2r$ or $2r + 1$. Hence *the degree and class are in general the same.*

§ 3. We shall confine our attention now to a quadratic system. The most general form may be written

$$(3) \quad (p\lambda^2 + 2q\lambda + r)S - (\lambda\alpha + \beta)^2 = 0,$$

where α and β represent distinct fixed lines, and p, q, r are given constants. If the lines α and β coincide, the equation of the system reduces to

$$(p\lambda^2 + 2q\lambda + r)S - (\lambda + k)^2\alpha^2 = 0.$$

For all values of λ this represents a system of circles with axis α , i. e., a concentric system. Though apparently a quadratic system, it is really linear, for two values of λ correspond to each circle.

The lines α and β may be replaced by any two distinct lines through their point of intersection without in any way altering the system; they might, for example, be the tangents to S , or a pair of conjugate lines with regard to S .

The axes of the circles form a pencil of lines $\lambda\alpha + \beta = 0$ through the fixed point $A \equiv (\alpha\beta)$, and the centers are collinear on the line a , the absolute polar of A .

§ 4. *Envelope of the system.* Writing the equation (3) in the form

$$\lambda^2(pS - \alpha^2) + 2\lambda(qS - \alpha\beta) + (rS - \beta^2) = 0,$$

the equation of the envelope is

$$(qS - \alpha\beta)^2 - (pS - \alpha^2)(rS - \beta^2) = 0,$$

i. e.,

$$(q^2 - pr)S^2 + (r\alpha^2 + p\beta^2 - 2q\alpha\beta)S = 0.$$

Hence the envelope consists of the absolute S and a *conic*

$$(4) \quad K \equiv (pr - q^2)S - (r\alpha^2 - 2q\alpha\beta + p\beta^2) = 0.$$

The equation (3) may be written in the form

$$(5) \quad (p\lambda^2 + 2q\lambda + r)K + \{(q\lambda + r)\alpha - (p\lambda + q)\beta\}^2 = 0,$$

which shows that *every circle of the system has double contact with the conic K , and the chords of contact all pass through A .*

§ 5. *Limiting lines.* In general there are two circles of the system which degenerate to coincident lines, limiting lines. The values of λ are the roots of the equation $p\lambda^2 + 2q\lambda + r = 0$, and the two lines are $r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0$. These form a pair of common chords of S and the conic envelope K , i. e., a pair of focal lines of K . Their point of intersection $A \equiv (\alpha\beta)$ is therefore a center of K , and its absolute polar a is an axis of K . Hence *the line of centers of the circles is one of the axes of the conic envelope, and the point of concurrence of the axes of the circles is the corresponding center of the conic envelope.* ■

The two limiting lines are a pair of focal lines of K ; the centers of these two degenerate circles are a pair of director points of K .

§ 6. *The pairs of corresponding chords of contact of the circles with S and K form an involution whose double lines are the limiting lines.*

To the line $\lambda\alpha + \beta = 0$ corresponds the line

$$(q\lambda + r)\alpha - (p\lambda + q)\beta = 0$$

or, say

$$\mu\alpha + \beta = 0,$$

so that

$$\mu(p\lambda + q) + (q\lambda + r) = 0,$$

i. e.,

$$p\lambda\mu + q(\lambda + \mu) + r = 0,$$

a symmetrical lineo-linear relation. The double elements are then given by

$$(6) \quad p\lambda^2 + 2q\lambda + r = 0,$$

the roots of which correspond to the two limiting lines.

If the roots of the equation (6) are c, d , and the limiting lines are γ, δ , the equation of the system of circles can be written

$$(\lambda - c)(\lambda - d)S - \{\gamma(\lambda - d) - \delta(\lambda - c)\}^2 = 0.$$

Then we find that

$$K \equiv S + 4\gamma\delta,$$

and the equation of the system can be written also

$$(\lambda - c)(\lambda - d)K - \{\gamma(\lambda - d) + \delta(\lambda - c)\}^2 = 0,$$

which verifies that the two chords of contact of any circle of the system with S and K are harmonic conjugates with regard to γ, δ .

From this symmetrical relationship it follows that if one circle has as chords of contact with S and K the lines u and v respectively, there is another circle of the system which has as chords of contact v and u respectively.

§ 7. *Limiting points.* S and K have four common tangents. Any two of these taken together form a degenerate circle having double contact with S and K . But of these pairs just one pair have their centers on the given line of centers of the system of circles. Hence *there are two circles of the system which degenerate to a line pair (point circles) and their centers form a pair of foci of the conic envelope K .* We shall call these two points the *limiting points* of the system.

§ 8. *Associated systems.* Let A, B, C be the three centers, a, b, c the corresponding axes of the conic K , and let $f, f'; g, g'; h, h'$ be the focal lines through A, B, C respectively; $F, F'; G, G'; H, H'$ the foci on a, b, c respectively. Then

there are three systems of circles, one with limiting points F, F' , and limiting lines f, f' , the second with limiting points G, G' , etc.

§ 9. *Horocycles.* In general the system contains two horocycles, corresponding to the two values of λ which make the line $\lambda\alpha + \beta = 0$ a tangent to S . These values of λ are the roots of the quadratic

$$\lambda^2(\Sigma \check{l}_1)^2 + 2\lambda(\Sigma \check{l}_1 \check{l}_2) + (\Sigma \check{l}_2)^2 = 0.$$

§ 10. *Degenerate systems.* The discriminant of the conic envelope

$$K \equiv (pr - q^2)S - (r\alpha^2 - 2q\alpha\beta + p\beta^2) = 0,$$

where

$$S \equiv (a_0 b_0 c_0 f_0 g_0 h_0 \check{x} y z)^2,$$

and

$$\alpha \equiv l_1 x + m_1 y + n_1 z, \quad \beta \equiv l_2 x + m_2 y + n_2 z,$$

is

$$(7) \quad (pr - q^2)^2 [\Delta_0(pr - q^2) - p(\Sigma \check{l}_2)^2 + 2q(\Sigma \check{l}_1)(l_2) - r(\Sigma \check{l}_1)^2 + (S \check{m}_1 n_2 - m_2 n_1)^2] = 0,$$

Δ_0 being the discriminant of S .

If $pr - q^2 = 0$, the equation of the conic envelope reduces to

$$r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0,$$

which now represents two coincident lines. As an envelope, the conic degenerates to two points. Put $p = k^2$, $q = k$, $r = 1$; then the equation of the system of circles (3) becomes

$$(k\lambda + 1)^2 S - (\lambda\alpha + \beta)^2 = 0,$$

i. e.,

$$\lambda^2(k^2 S - \alpha^2) + 2\lambda(kS - \alpha\beta) + (S - \beta^2) = 0.$$

Now

$$k^2 S - \alpha^2 \equiv k^2(S - \beta^2) + (k^2 \beta^2 - \alpha^2),$$

and

$$kS - \alpha\beta \equiv k(S - \beta^2) + \beta(k\beta - \alpha).$$

Hence the three conics $k^2 S - \alpha^2 = 0$, $kS - \alpha\beta = 0$, $S - \beta^2 = 0$ have two points in common, their common chord being $\alpha - k\beta = 0$. All the circles of the system therefore pass through these two fixed points, and this point pair is the

degenerate conic envelope K . We have then a *coaxal system* of circles.

When the other factor of the discriminant (7) vanishes, the conic envelope reduces to two distinct lines, and we have a *homocentric system* of circles with two common tangents. Multiplying the factor by Δ_0 , the condition can be expressed in the form

$$[\Delta_0 p - (\Sigma \check{l}_1)^2][\Delta_0 r - (\Sigma \check{l}_2)^2] - [\Delta_0 q - (\Sigma \check{l}_1 \check{l}_2)]^2 = 0.$$

§ 11. The system will be specialized also if the conic envelope is a circle. If K is a circle the common chords $r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0$ must either coincide or be tangents to the absolute. In the former case, however, K degenerates to these two coincident straight lines, or, as an envelope, to a point pair.

The equation of the tangents from the point $(\alpha\beta)$ to the absolute is

$$\alpha^2(\Sigma \check{l}_2)^2 - 2\alpha\beta(\Sigma \check{l}_1 \check{l}_2) + \beta^2(\Sigma \check{l}_1)^2 = 0.$$

Hence the conditions that K should be a circle are

$$p = k(\Sigma \check{l}_1)^2, \quad q = k(\Sigma \check{l}_1 \check{l}_2), \quad r = k(\Sigma \check{l}_2)^2.$$

Then

$$pr - q^2 = k^2 \Delta_0 (S \check{m}_1 n_2 - m_2 n_1)^2,$$

and the discriminant of K , (7), becomes

$$k^4 \Delta_0^2 [(S \check{m}_1 n_2 - m_2 n_1)^2]^3 (k \Delta_0 - 1)^2.$$

In this case therefore K does not in general degenerate further unless the point $(\alpha\beta)$ lies on the absolute. Hence *the conic envelope may be a circle, but cannot be a horocycle*.

When K is a circle its axis is the line of centers of the circles of the system. The common chords with S and K , being harmonic conjugates with regard to the two common tangents of S and K , are at right angles. The chords of contact with K all pass through $(\alpha\beta)$ and are therefore perpendicular to the axis of K . Hence all the circles of the system have equal radii. The system therefore consists of a system of equal circles with collinear centers. The limiting points reduce to the two absolute points of K , and the limiting lines to the absolute lines of K .

§12. Taking two conjugate lines through $(\alpha\beta)$ and the absolute polar of $(\alpha\beta)$ as the triangle of reference, the equation of the general quadratic system of circles can be written

$$(p\lambda^2 + 2q\lambda + r)(x^2 + y^2 + z^2) - (\lambda x + y)^2 = 0.$$

Let the radius be ρ , then

$$(8) \quad c^2 = \cos^2 \rho = \frac{(\lambda x + y)^2}{(x^2 + y^2 + z^2)(\lambda^2 + 1)} = \frac{p\lambda^2 + 2q\lambda + r}{\lambda^2 + 1}.$$

Hence *there are in general two circles of the system with a given radius.*

Let the parameters of the two circles be λ_1 and λ_2 , which are the roots of the equation

$$(9) \quad (p - c^2)\lambda^2 + 2q\lambda + (r - c^2) = 0.$$

Then

$$\lambda_1 + \lambda_2 = -2q/(p - c^2),$$

$$\lambda_1\lambda_2 = (r - c^2)/(p - c^2).$$

Eliminating c^2 , we get

$$q\lambda_1\lambda_2 - \frac{1}{2}(p - r)(\lambda_1 + \lambda_2) - q = 0.$$

Hence *the parameters of pairs of equal circles are connected by an involutorial homographic relation.*

There are *two coincident pairs of equal circles*, whose parameters are the roots of the equation

$$q\lambda^2 - (p - r)\lambda - q = 0,$$

and it is easily verified that these are the circles of maximum or minimum radius. The radii of these two circles are found by expressing the condition for equal roots in the equation for λ , (9); we get then the equation

$$c^4 - (p + r)c^2 + (pr - q^2) = 0.$$

if these two values of the radius are equal, all the radii will be equal. The conditions for this are $q = 0$ and $p = r$. Equation (8) cannot then be satisfied except for a particular value of c .

If there are three circles of the system with equal radii, all the circles of the system must have equal radii.

§ 13. *Two-parameter system of circles.* There is no linear two-parameter system of circles. The general quadratic system can be written in terms of the homogeneous parameters λ, μ, ν

$$(10) \quad (a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu)S \\ - (\lambda\alpha + \mu\beta + \nu\gamma)^2 = 0,$$

where α, β, γ represent three straight lines.

If the lines α, β, γ are concurrent, so that $\gamma \equiv l\alpha + m\beta$, then, putting $\lambda + l\nu = \lambda', \mu + m\nu = \mu'$, the equation can be reduced to the form

$$(a'\lambda'^2 + \dots + 2f'\mu'\nu' + \dots)S - (\lambda'\alpha + \mu'\beta)^2 = 0.$$

This represents a system of circles in which the only restriction is that the center lies on a fixed straight line, the polar of $(\alpha\beta)$.

If α, β, γ are coincident the circles are concentric with center the pole of α .

Passing to the general case in which α, β, γ form a triangle, we can transform the equation by a linear transformation so that the triangle α, β, γ is transformed into a self-conjugate triangle of the absolute. Taking this as triangle of reference the equation can be written in the form

$$(11) \quad P_1S - (\lambda x + \mu y + \nu z)^2 = 0,$$

where

$$S \equiv x^2 + y^2 + z^2,$$

and

$$P_1 \equiv a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu.$$

The center of the circle is therefore (λ, μ, ν) . When $\lambda : \mu : \nu$ are given, the circle is fixed, hence, *with a given point as center there is one and only one circle of the system.*

The line equation of the circle (11) is found to be

$$(\lambda^2 + \mu^2 + \nu^2 - P_1)(\xi^2 + \eta^2 + \zeta^2) - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0,$$

or

$$(12) \quad P_0\Sigma - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0.$$

§ 14. *Locus of limiting points and envelope of limiting lines.*
The discriminant of (11) is

$$\begin{vmatrix} P_1 - \lambda^2 & -\mu\lambda & -\nu\lambda \\ -\lambda\mu & P_1 - \mu^2 & -\nu\mu \\ -\lambda\nu & -\mu\nu & P_1 - \nu^2 \end{vmatrix} = 0$$

i. e.,

$$P_1^2(P_1 - \lambda^2 - \mu^2 - \nu^2) = 0 \quad \text{or} \quad P_1^2 P_0 = 0.$$

If $P_1 = 0$ the circle (11) degenerates to two coincident lines $\lambda x + \mu y + \nu z = 0$. The line equation of the envelope of these limiting lines is therefore

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0.$$

The point equation of the envelope of limiting lines is therefore

$$(13) \quad S_1' \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

The locus of the absolute poles of these lines, or centers of the circles which degenerate to coincident lines, is the reciprocal of S_1' with regard to S , and its point equation is

$$(14) \quad S_1 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The circle (12) degenerates to two distinct straight lines, or a point circle, if $P_0 = 0$. The center being (λ, μ, ν) , the equation of the locus of point circles is therefore

$$(15) \quad S_0 \equiv S - S_1 = 0.$$

The envelope of the axes of the point circles is the reciprocal of (15) with regard to S , and its point equation is thus found to be

$$(16) \quad S_0' \equiv S_1 + S_1' + (1 - \Sigma a)S = 0.$$

§ 15. *Horocycles.* The circle will be a horocycle if the line $\lambda x + \mu y + \nu z = 0$ is a tangent to S , i. e., if

$$\lambda^2 + \mu^2 + \nu^2 = 0.$$

Hence through any point there pass four horocycles of the system.

§ 16. Consider the circles of the system whose centers lie on a given line $lx + my + nz = 0$. We shall call this a *row of circles*.

We have

$$l\lambda = m\mu + n\nu = 0.$$

Eliminating ν , we get

$$[n^2(a\lambda^2 + 2h\lambda\mu + b\mu^2) - 2n(g\lambda + f\mu)(l\lambda + m\mu) + c(l\lambda + m\mu)^2]S - [n(\lambda x + \mu y) - (l\lambda + m\mu)z]^2 = 0.$$

This is a one-parameter system of circles

$$(17) \quad [(an^2 - 2gnl + cl^2)\lambda^2 + 2(hn^2 - gmn - fnl + clm)\lambda\mu + (bn^2 - 2fmn + cm^2)\mu^2]S - [(nx - lz)\lambda + (ny - mz)\mu]^2 = 0,$$

having as envelope the conic

$$(18) \quad T \equiv (Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm)S - \Sigma x^2(cm^2 - 2fmn + bn^2) + 2\Sigma yz(fl^2 + amn - hnl - glm) = 0.$$

This is a two-parameter quadratic system of conics. Two of the foci of T are limiting points of the system of circles and therefore lie on the conic S_0 , and the corresponding directrices touch the conic S_0' ; the two corresponding focal lines are limiting lines of the system and therefore touch the conic S_1' , and the corresponding director points lie on the conic S_1 .

§ 17. The two lines

$$\Sigma x^2(cm^2 - 2fmn + bn^2) - 2\Sigma yz(fl^2 + qmn - hnl - glm) = 0,$$

which pass through the point of intersection of $nx - lz = 0$ and $ny - mz = 0$, i. e., the point (l, m, n) , are in fact the tangents from this point to the conic S_1' , and the equation of the conic envelope can be written

$$\Delta T \equiv (Al^2 + \dots + 2Fmn + \dots)(\Delta S - S_1') - [x(Al + Hm + Gn) + \dots]^2 = 0.$$

Hence the conic envelopes T all have double contact with the fixed conic $\Delta S - S_1' = 0$.

If the conic S' coincides with the absolute S , the conics T are all circles. In this case all the circles of the system have equal radii.

We shall consider in the next section what rows of circles with equal radii exist in the general system.

§ 18. Let ρ be the radius of the circle

$$P_1(x^2 + y^2 + z^2) - (\lambda x + \mu y + \nu z)^2 = 0.$$

Then

$$\cos^2 \rho = \frac{(\lambda x + \mu y + \nu z)^2}{(x^2 + y^2 + z^2)(\lambda^2 + \mu^2 + \nu^2)} = \frac{P_1}{\lambda^2 + \mu^2 + \nu^2}.$$

If the center (λ, μ, ν) is given, the radius is in general determined, so that with a given center there is in general just one circle of the system. ρ will, however, be indeterminate if both

$$P_1 = 0 \quad \text{and} \quad \lambda^2 + \mu^2 + \nu^2 = 0.$$

Hence if the center is at one of the four absolute points on the conic S_1 or S_0 the radius is indeterminate.

Consider the locus of centers of circles with a given radius. The equation of the locus is

$$(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) - (x^2 + y^2 + z^2) \cos^2 \rho = 0.$$

Hence the locus is a conic homothetic with S_1 and S_0 .

Putting $\cos^2 \rho = r$, the values of r for which this conic breaks up into a pair of straight lines are the roots of the equation

$$\begin{vmatrix} a - r & h & g \\ h & b - r & f \\ g & f & c - r \end{vmatrix} = 0$$

This equation has three real roots, hence *there are three pairs of rows of circles each with equal radii*. We shall call the lines of centers of these circles the *equiradial axes*.

The three pairs of equiradial axes are common chords of S and S_1 , i. e., the pairs of focal lines of the conic S_1 (or S_0). The axes of these rows of circles are concurrent in three pairs of points which are the director points of S_1 (or S_0), or the foci of S_1' (or S_0').

These results are easily proved also directly by taking as the triangle of reference the common self-conjugate triangle of S and S_1 , so that the equation of the system becomes

$$(a\lambda^2 + b\mu^2 + c\nu^2)(x^2 + y^2 + z^2) - (\lambda x + \mu y + \nu z)^2 = 0.$$

The three pairs of equiradial axes are then given by

$$(a - b)y^2 - (c - a)z^2 = 0,$$

$$(b - c)z^2 - (a - b)x^2 = 0,$$

$$(c - a)x^2 - (b - c)y^2 = 0,$$

the radii being respectively $\cos^{-1} \nu a$, $\cos^{-1} \nu b$, $\cos^{-1} \nu c$.

If $b = c$, the conics S_1 and S_0 have double contact with S and are concentric circles. In this case the last two pairs of axes coincide with $x = 0$, which is the axis of S_1 or S_0 , and the first pair reduces to $y^2 + z^2 = 0$, which consists of the absolute tangents of S_1 or S_0 .

§ 19. The discriminant of the conic envelope T is

$$\Sigma_1^2 \Sigma_0 = 0,$$

where

$$\Sigma_1 \equiv \Sigma A l^2 + 2 \Sigma F m n,$$

$$\Sigma_0 \equiv (\Sigma A l^2 + 2 \Sigma F m n) + (\Sigma a l^2 + 2 \Sigma f m n) - \Sigma l^2 (\Sigma a - 1).$$

When $\Sigma_1 = 0$, T degenerates to a point pair. Since $\Sigma_1 = 0$ is the line equation of the conic S_1 , we see that *the circles whose centers all lie on a tangent to S_1 form a coaxial system*, the common radical axis being the absolute polar of the point of contact of this tangent.

When $\Sigma_0 = 0$, T degenerates to a line pair. Since $\Sigma_0 = 0$ is the line equation of the conic S_0 , we see that *the circles whose centers all lie on a tangent to S_0 form a homocentric system*, the common homothetic center being the point of contact of this tangent.

§ 20. T will degenerate to a point pair for every row of circles if Σ_1 vanishes identically, i. e., if P_1 is a perfect square. Then all the limiting lines pass through one point, and the locus of point circles S_0 becomes a circle with this point as center. In this case we can prove that *every circle of the system cuts the circle S_0 orthogonally*.

Let $a = p^2$, $b = q^2$, $c = r^2$, $f = qr$, $g = rp$, $h = pq$, then

$$\begin{aligned} -T &\equiv \Sigma x^2 (rm - qn)^2 + 2 \Sigma yz (pn - rl)(ql - pm) \\ &\equiv [\Sigma (rm - qn)x]^2. \end{aligned}$$

Hence the point pair lies on the line $\Sigma (rm - qn)x = 0$ which

passes through the fixed point (p, q, r) , the center of the circle S_0 .

The polar of (p, q, r) with regard to the circle

$$(\Sigma p\lambda)^2 \cdot \Sigma x^2 - (\Sigma \lambda x)^2 = 0$$

is

$$\Sigma p\lambda \cdot \Sigma px - \Sigma \lambda x = 0.$$

Now forming the equation of a pair of common chords of this circle and the circle

$$S_0 \equiv (\Sigma px)^2 - \Sigma x^2,$$

we have

$$\begin{aligned} (\Sigma p\lambda)^2 \cdot \Sigma x^2 - (\Sigma \lambda x)^2 + [(\Sigma px)^2 - \Sigma x^2](\Sigma p\lambda)^2 \\ = (\Sigma px)^2(\Sigma p\lambda)^2 - (\Sigma \lambda x)^2. \end{aligned}$$

Hence one of the common chords is $\Sigma px \cdot \Sigma p\lambda - \Sigma \lambda x = 0$, which is the polar of (p, q, r) . Hence the circles cut orthogonally.

§ 21. T will degenerate to a line pair for every row of circles of the system if Σ_0 vanishes identically. This is the condition that P_0 should be a perfect square.

Now every point on a circle is tangentially distant a quadrant from its axis, and if two circles are such that the length of one of their common tangents is a quadrant, say the circles are *quadrantal*, each of the points of contact of this tangent must lie on the axis of the other circle. The same will be true for the other common tangent belonging to the same pair. If these two tangents intersect in O , the polar of O with regard to each circle is the axis of the other circle.

Since the line equation of the system of circles is

$$P_0\Sigma - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0,$$

the analytical work in deducing the conclusions from the conditions that P_0 should be a perfect square is exactly the same as that of the last section. Hence if this condition is satisfied, *the locus of point circles consists of two coincident straight lines, the envelop of limiting lines is a circle, of which the locus of point circles is the axis, and every circle of the system is quadrantal with respect to this circle.*

VICTORIA UNIVERSITY COLLEGE,
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CONTINUOUS SETS THAT HAVE NO CONTINUOUS SETS OF CONDENSATION.

BY PROFESSOR R. L. MOORE.

(Read before the American Mathematical Society, October 27, 1917.)

JANISZEWSKI has shown* that if A and B are two distinct points then every bounded set of points that is *irreducibly continuous*† from A to B , and has no continuous set of condensation,‡ is a simple continuous arc from A to B . In the present paper I will establish the following result.

THEOREM. *Every bounded continuous set of points that has no continuous set of condensation is a continuous curve.‡*

Proof. Suppose M is a bounded continuous set of points that has no continuous set of condensation. It has been shown by Hahn§ that every bounded continuous set of points that is connected “im kleinen” is a continuous curve. I shall proceed to show that the point set M is connected “im kleinen.” Suppose that it is not. Then there is a point P belonging to M and a circle K with center at P such that within every circle whose center is P there exists a point which does not lie together with P in any connected subset of M that lies

* S. Janiszewski, “Sur les continus irréductibles entre deux points,” *Journal de l'Ecole Polytechnique*, 2e série, vol. 16 (1911–12), pp. 79–170.

† A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of these subsets contains a limit point of the other one. A set of points is said to be *continuous* if it is closed and connected and contains more than one point. A continuous set of points containing the two distinct points A and B is said to be *irreducibly continuous* from A to B if it contains no other continuous set that contains both A and B . The continuous set N is said to be a *continuous set of condensation* of the continuous set M if N is a subset of M and every point of N is a limit point of $M-N$.

‡ A *continuous curve* is the set of all points $\{(x, y)\}$ satisfying the equations $x = f_1(t)$, $y = f_2(t)$ ($0 \leq t \leq 1$), where $f_1(t)$ and $f_2(t)$ are continuous functions of t . In case there do not exist, in the interval ($0 \leq t \leq 1$), two distinct numbers t_1 and t_2 such that $f_1(t_1) = f_1(t_2)$ and $f_2(t_1) = f_2(t_2)$, then this curve is a *simple continuous arc*.

§ Hans Hahn, “Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist,” *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318–322. A set of points M is said to be *connected “im kleinen”* (cf. Hahn, loc. cit.) if for each point P of M and each circle K with center at P there exists, within K , another circle K' , with center at P , such that if X is a point of M within K' then X and P lie together in some connected subset of M that lies entirely within K .

entirely within K . Let $K_1, K_2, K_3, \dots$ denote an infinite sequence of circles with center at P such that the radius of K_n is $r/2n$, where r is the radius of K . For every n there exists, within K_n , a point X_n such that X_n and P do not lie together in any connected subset of M that lies entirely within K . Let $\bar{K}$ denote a definite circle, with center at P , that lies within K and encloses K_1 . By a theorem due to Janiszewski,* M contains at least one point set that is irreducibly continuous from X_n to P . Let t_n denote one such set. The set t_n is a subset of M and M has no continuous set of condensation. It follows that t_n has no continuous set of condensation. Hence, by the first of the above mentioned theorems of Janiszewski, t_n is a simple continuous arc† from X_n to P . This arc can not lie entirely within K . Let Y_n denote the first point, in the order from X_n to P , that it has in common with $\bar{K}$. The interval $X_n Y_n$ of the arc t_n lies, except for the point Y_n , entirely within $\bar{K}$. There exists n_1 such that if $n \geq n_1$ then $X_n Y_n$ has no point in common with $X_1 Y_1$. For otherwise there would exist an infinite subsequence $\bar{X}_1 \bar{Y}_1, \bar{X}_2 \bar{Y}_2, \bar{X}_3 \bar{Y}_3, \dots$ of the sequence $X_1 Y_1, X_2 Y_2, X_3 Y_3, \dots$ such that, for every n , $\bar{X}_n \bar{Y}_n$ has a point in common with $X_1 Y_1$. The point set composed of the point P together with the sum of the arcs $X_1 Y_1, \bar{X}_1 \bar{Y}_1, \bar{X}_2 \bar{Y}_2, \bar{X}_3 \bar{Y}_3, \dots$ would then be a connected point set lying within K and containing both P and X_1 . Thus a contradiction would be obtained. Similarly there exists n_2 , greater than n_1 , such that if $n \geq n_2$ then $X_n Y_n$ has no point in common with $X_{n_1} Y_{n_1}$. There exists n_3 greater than n_2 such that if $n > n_3$ then $X_n Y_n$ has no point in common

* S. Janiszewski, "Sur la géométrie de lignes cantorienes," *Comptes Rendus*, vol. 151 (1910), pp. 198–201.

† It may be of interest to note that in order that a bounded continuous set of points M should be a continuous curve it is not sufficient that every two points of M should be the extremities of a simple continuous arc lying wholly in M . To see this consider the following example.

Example. Let AB denote the interval from $(0, 0)$ to $(0, 1)$, in a rectangular system of coordinates, and let $B_1, B_2, B_3, \dots$ denote the points $(1, 1), (1, \frac{1}{2}), (1, \frac{1}{3}), \dots$ respectively. The point set composed of the intervals $AB, AB_1, AB_2, AB_3, \dots$ is a bounded, continuous set of points M every two points of which can be joined by an arc that lies in M . But M is not connected "im kleinen" and is therefore not a continuous curve.

However in my paper "A theorem concerning continuous curves," this BULLETIN, vol. 23 (1917), pp. 233–236, I proved the truth of the converse proposition that every two points of a continuous curve are the extremities of at least one simple continuous arc lying in the point set constituted by that curve.

with $X_{n_i}Y_{n_i}$. If this process is continued there will be obtained an infinite sequence of arcs $X_{n_1}Y_{n_1}, X_{n_2}Y_{n_2}, X_{n_3}Y_{n_3}, \dots$ no two of which have any point in common. For each i , the arc $X_{n_i}Y_{n_i}$ contains, as a subset, an arc $W_{n_i}Y_{n_i}$ which lies between the circles $\bar{K}$ and K_1 , except for the points W_{n_i} and Y_{n_i} which lie on K_1 and $\bar{K}$ respectively. There exist 1) on $\bar{K}$ an infinite sequence of distinct points $Y', Y_1', Y_2', Y_3', \dots$, 2) on K_1 an infinite sequence of distinct points $W', W_1', W_2', W_3', \dots$, 3) an infinite sequence of distinct arcs $W_1'Y_1', W_2'Y_2', W_3'Y_3', \dots$ all belonging to the set $W_{n_1}Y_{n_1}, W_{n_2}Y_{n_2}, W_{n_3}Y_{n_3}, \dots$, such that Y' is the sequential limit point of the sequence $Y_1', Y_2', Y_3', \dots$ and W' is the sequential limit point of the sequence $W_1', W_2', W_3', \dots$. No two of the arcs $W_1'Y_1', W_2'Y_2', W_3'Y_3', \dots$ have a point in common. It easily follows that there exists a closed connected point set N , containing Y' and W' , such that every point of N is a limit point of the point set constituted by the sum of the arcs $W_1'Y_1', W_2'Y_2', W_3'Y_3', \dots$. The point set N is a continuous set of condensation of the set M .

Thus the supposition that M is not connected "im kleinen" leads to a contradiction. It follows that M is a continuous curve.

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DERIVATIVELESS CONTINUOUS FUNCTIONS.

BY PROFESSOR M. B. PORTER.

(Read before the American Mathematical Society October 26, 1918.)

THERE is no more interesting illustration of the refinement of geometric intuition through the influence of the arithmetization of mathematics than that presented by the history of functions of this type. No less a mathematician than Ampère, not to mention Duhamel and Bertrand, thought he had actually proved that continuous functions had derivatives for all save a finite number of arguments. Darboux in his paper on "Discontinuous functions" published in the *Annals* of the Ecole Normale for 1875, though dated January 20, 1874, in

connection with his example of a function of this class mentions only one auditor, M. Bienaimé, who said that he was unconvinced by Ampère's proof. Darboux makes no mention of Weierstrass's work published in 1874 in *Crelle* on the function $\sum a^n \cos b^n x$ and was doubtless not aware of it. Du Bois-Reymond was so awestruck by Weierstrass's curve as to provoke some rather jocular remarks by Wiener in the introduction to his paper on Weierstrass's curve in the 90th volume of *Crelle's Journal*.

It does not seem to have been noticed by writers who have considered these functions, especially of Weierstrass's type, that a considerable simplification of treatment was possible by a more obvious choice of the δx used in the incremental ratio and that other advantages might result from such an innovation.

We begin by treating in detail a Weierstrass function

$$W(x) = \sum_0^{\infty} a^n \sin b^n \pi x \quad |a| < 1, b \text{ integral.}$$

Setting

$$\delta x = 2k/b^{N+1}, \quad k \text{ integral,}$$

we get by applying the mean value theorem to the first N terms and a trigonometric identity to the last term,

$$(1) \quad \frac{\Delta W(x)}{\delta x} = \pi \sum_0^{N-1} (ab)^n \cos b^n \pi (x + \theta \delta x) \\ + \pi (ab)^N \sin \frac{k\pi}{\pi k/b} \cos \left(b^N x + \frac{k}{b} \right) \pi.$$

Evidently the absolute value of the first N terms is less than

$$\pi \sum_0^{N-1} |ab|^n < \frac{\pi |ab|^N}{|ab| - 1}$$

if $|ab| > 1$. As to the last term in (1), if we suppose that $k \nmid \frac{3}{4}b$, which implies that 4 is a factor of b , it is, in absolute value,

$$(2) \quad \geq |\pi |ab|^N \frac{1}{\frac{3}{4}\pi \sqrt{2}} \left| \cos \left(b^N x + \frac{k}{b} \right) \pi \right|$$

In this expression we can find two values of k , k_1 and k_2 , such that

$$1^\circ. \quad 1 \geq \cos \left(b^N x + \frac{k_1}{b} \right) \pi \geq \frac{1}{\sqrt{2}}$$

$$2^\circ. \quad -\frac{1}{\sqrt{2}} \geq \cos \left(b^N x + \frac{k_2}{b} \right) \pi \geq -1$$

where k_1 and k_2 have *opposite** signs.

Thus for these values of k

$$\left| \pi(ab)^N \frac{1}{\frac{3}{4}\pi\sqrt{2}} \cos \left(b^N x + \frac{k}{b} \right) \pi \right| \geq \frac{\pi|ab|^N}{\frac{3}{2}\pi}.$$

The last term of (1) will thus dominate in sign and magnitude the first N terms if

$$|ab| > 1 + \frac{3}{2}\pi.$$

Hence the right and left incremental ratio which we are considering will become infinite with N but will always have opposite signs.

This proves that $W(x)$ has a derivative for no value of x .

Making use of the remark at the end of the footnote, we can prove that, except possibly for a set of x -points of Borel measure zero (null set), $W(x)$ has neither a backward nor a forward tangent.

To do this suppose x written in system radix b , i. e.,

$$x = b_0 + \frac{b_1}{b} + \frac{b_2}{b^2} + \dots,$$

where b_1, b_2 , etc., $< b$. All those x 's in which any one of the digits $0, 1, \dots, b-1$ fails to occur infinitely often form a null set. Hence, except for the points of the null set $[x]$

* This can be seen at once from a diagram. Divide the angle 2π into 8 equal parts. If for example $b^N x \pi$ lies in the first octant, one and possibly more values of k can be found between 1 and $\frac{1}{4}b$ so that 1° holds; while one or possibly more values lie between 0 and $-\frac{1}{4}b$ for which 2° holds. Similarly for the other octants.

Moreover it is clear that if $b^N x \pi$ has its terminal line between $I\pi$ and $(I + 1/b)\pi$, I integral or zero, a positive k_1 can be found for which 1° holds and another positive k_2 for which 2° holds. If $b^N x \pi$ lies between $(I + (b-1)/b)\pi$ and $(I + 1)\pi$, two negative k 's can be found for which 1° and 2° hold respectively.

for which $b_i = 0$ and $b_i = b - 1$ fail to occur infinitely often, the right and left incremental ratios have each for their upper and lower limits $+\infty$ and $-\infty$ respectively. The set $[x]$ is made up of a countable number of perfect null sets and is everywhere dense on the line; it is consequently not closed.

At the points of $[x]$ $W(x)$ might possibly have vertical cusps but not elsewhere.

It will be noted that the signs of the individual terms of our series are of no significance and hence can be arbitrarily changed to $+$ or $-$. This conclusion does not seem to follow from Weierstrass's proof.

Weierstrass (also Dini and others) unnecessarily require that b be an odd integer; our restriction that b be a multiple of 4 can evidently be removed if $|ab| > 9$, and the proof just given will remain valid.

The formation of an extensive class of such functions

$$W(x) = \sum_0^{\infty} u_n(x) \sin i_n x \pi \quad \text{or} \quad \sum_0^{\infty} u_n(x) \cos i_n x \pi,$$

i_n integral, is easy. We require:

1°. Uniform convergence when $l \leq x \leq L$.

2°. i_n must divide i_{n+1} and for an unlimited number of n 's their ratio must either be divisible by 4 or increase indefinitely with n .

3°. $\sum_0^{\infty} u_n'(x)$ must be uniformly convergent by Weierstrass's G test, $l \leq x \leq L$.

4°. $\frac{3}{2}\pi \sum_0^{N-1} |i_n u_n(x)| < |i_N u_N(x)|$ in the interval of convergence of $W(x)$.

These conditions are merely sufficient and 2° and 4° admit of certain obvious modifications. Functions which fulfill these conditions will have no derivatives in the interval (lL) .

The following specimens will suffice:

$$1. \sum_0^{\infty} \frac{a^n \sin}{n! \cos} (n! \pi x) \quad |a| > 1 + \frac{3}{2}\pi.$$

2. $\sum_1^{\infty} \frac{a^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cos (1 \cdot 3 \cdot 5 \cdots 2n-1)\pi x$
 $|a| > 1 + \frac{3}{2}\pi$ (Dini).
3. $\sum_1^{\infty} \frac{a^n}{1 \cdot 5 \cdot 9 \cdots (4n+1)} \sin (1 \cdot 5 \cdot 9 \cdots (4n+1))\pi x$
 $a > 1 + \frac{3}{2}\pi$ (Dini).
4. If $\sum_1^{\infty} \frac{a_i}{10^i}$ denote any non-terminating decimal,

$$\sum_0^{\infty} \frac{a_i}{10^i} \frac{\sin}{\cos} (10^{3i}x\pi).$$
5. $\sum_0^{\infty} \frac{1}{a^n} \frac{\sin}{\cos} (n!a^n\pi x)$, where $|a|$ is an integer > 1 .
6. $x \sum_0^{\infty} a^n \sin b^n x\pi$, $|a| < 1$, $|ab| > 1 + \frac{3}{2}\pi$, has a derivative
for $x = 0$ but for no other value of x .
7. $\sum_0^{\infty} \frac{x^n}{n!} \frac{\sin}{\cos} (n!\pi x)$ has derivatives *between* -1 and $+1$ and
no derivatives if $|x| > 1 + \frac{3}{2}\pi$.

Lerch gives a theorem* which shows that this last function has no derivatives for any *rational* points for which $|x| \geq 1$. It is easy to show that it can have a finite derivative for no point $|x| > 1 + \frac{\pi}{2}$.

A HALF CENTURY OF FRENCH MATHEMATICS.

Les Sciences Mathématiques en France depuis un Demi-Siècle.

Par EMILE PICARD. Paris, Gauthier-Villars, 1917. 24 pp.

In the first decades of the last century the home of the scientific spirit was in France. Paris was the capital of the Republic of exact truth. Interest in scientific discovery and creation was widespread among her people. The spirit of literature flourished alongside the spirit of exact researches

* Lerch, *Crelle's Journal*, vol. 103, p. 130 ("Ueber die Nichtdifferentiierbarkeit gewisser Funktionen").

and both found place in the same creative intellect. Out of this union of elements, too much separated in other countries, there grew up a tradition of literary excellence in scientific exposition which abides to the present and contributes in no small way to the comfort and delight which every one must feel in reading a French scientific book or memoir.

From the extreme precision and abstract character of mathematics it probably does not lend itself as readily to literary form as most other sciences; but, even here, the French have found a method of presentation which renders delightful both the books and memoirs of her masters. On account of their preëminence in this respect it is probably this which first comes to mind when one thinks of the last half century of French mathematics.

Yet the actual and detailed contributions of French investigators to the body of recent discoveries in mathematics looms large in quantity and bears the distinguishing mark of the highest quality. Indeed, leading mastery in the art of presentation can grow out of nothing other than a penetrating insight into the secrets of a subject.

Judged in the light of the excellence of French exposition that which is current in America is seen to fall well below the standard to which we should seek to attain. Among us there is a dearth of expository treatments, both of the sort by which one may get his introduction to a chapter of modern mathematics and of that by which he may obtain a well-rounded initial view of a comprehensive subject. Our memoirs are often printed in such condensed form that even practised mathematicians must waste in reading them time which might have been saved by the use of larger space in setting forth a more illuminating exposition. There is also a tendency towards the publication of matter which is not sufficiently well digested in thought. Results at the half-way stage towards the goal of an investigation are harder to understand than those of the ultimate reach which come of a more penetrating analysis. Too many memoirs appear profound because the writers have not discovered the simpler and more fundamental truths. The French are at present the leading masters in avoiding this defect and attaining the opposite virtue.

The production of work of great value requires extended continuity of spirit and effort. In recent French mathematics

this truth is illustrated by the intimate relation of later discoveries to the earlier work of Fourier, Cauchy, and Galois. A profound study of nature was the source of the fruitful mathematical discoveries of Fourier, who found in physics the first origins of his great analytical theories and who developed analysis primarily for its usefulness to him as a physicist. His work, admirable for its clarity, contained the germ of important methods since employed in the fuller development of the theory of differential equations and of the expansion of arbitrary functions. At the opposite extreme was Galois with his investigation of the theory of algebraic equations and their groups. His researches were far removed from practical experience and even from contact with any theory of nature. He was engaged in problems belonging to the most abstract of pure mathematical disciplines. Between these two and reaching to both extremes was found the prodigious activity of Cauchy, whose researches extended to every domain of pure and applied mathematics. His greatest creation was that of the theory of functions of a complex variable.

During the past hundred years—and no less in its latter half than in its former—the theory of functions of a complex variable has been in great honor among the French. To an extent unusual in our generation we have here a vast subject dominated by the spirit of a single people. It is significant of the character of French thought that this theory of functions of a complex variable is perhaps the most elegant theory of like extent in the whole domain of mathematics.

The pamphlet under review is reprinted from a work published in 1916 with the title *Un Demi-Siècle de Civilisation française (1870–1915)* and thus covering the period between the present and the last great war fought by France. After an introduction of three pages, Picard gives here a brief analysis of the French mathematics of the period, presenting his matter under the following headings: analytic functions (pages 4–9); differential equations (pages 9–14); theory of numbers, algebra and geometry (pages 14–19); theory of functions of real variables and theory of assemblages (pages 19–23); some final remarks (pages 23–24).

During the last fifty years an important part of French mathematical effort has been devoted to the theory of functions both in its general aspects and as regards certain special classes of functions. Among those who have been interested

in the general theory we may cite the names of Laguerre, Poincaré, Picard, Appell, Goursat, Painlevé, Hadamard, Borel. The demonstration of the Cauchy theorem concerning the zero value of an integral along a contour had formerly assumed the continuity of the derivative; Goursat has shown that this hypothesis is unnecessary. Picard has established the fact that in the neighborhood of an isolated essential singularity a function takes infinitely often every given value with the possible exception of at most *two* values; many authors have drawn various consequences from this remarkable theorem. We may also call attention to the developments, in series of polynomials, of Appell and Painlevé, to the functions with lacunary spaces of Poincaré and Goursat, and to the more recent developments of Montel. Concerning the fundamental character of a power series on the circle of convergence are to be found Hadamard's classic and far-reaching investigations. Laguerre's study of entire functions, Borel's treatment of summable series, and Poincaré's work on multiform functions are other matters to which attention should be directed.

In our period the most important work dealing with special classes of functions has been that of Poincaré on automorphic functions. The fuchsian functions afford a means of uniform parametric representation of any algebraic curve, "certainly one of the most profound results obtained in analysis in the last fifty years."

In recent years Borel has made a penetrating study and comparison of the notion of *analyticity* in the sense of Weierstrass and of that of *monogeneity* in the sense of Cauchy, and has reached the conclusion that the latter is the essential one. The detailed investigations on which this conclusion is based are set forth in Borel's *Leçons sur les Fonctions monogènes uniformes*, 1917. The essential results, however, were announced at the Cambridge Congress in 1912.

In the hands of Picard the method of successive approximation has been a useful tool in the study of differential equations. Poincaré has introduced the general notion of asymptotic representation for the solutions of difference equations and of differential equations in the neighborhood of an irregular singular point. Poincaré has shown how linear algebraic differential equations may be integrated by means of thetáfuchsian series. Painlevé has done fundamental work in the classification of non-linear differential equations with

fixed critical points. Poincaré has devoted several memoirs to the study of curves defined by differential equations and has made particularly notable contributions to the theory of the differential equations of physics. The work of Darboux on differential geometry is everywhere recognized as of leading importance. The same author has also created a new method of integrating partial differential equations; Goursat has devoted several papers to elucidating the questions suggested by this method.

The methods created by Hermite have given to the theory of numbers a new and broader horizon. The work of Jordan on the theory of forms has realized important progress in the general theory of algebraic forms of higher degree. Poincaré has left his mark on the theory of quadratic forms. Jordan has made a profound study of the epoch-making ideas of Galois. Halphen has extended the theory of twisted algebraic curves. Cartan has contributed to our knowledge of the structure of groups and to the determination of simple groups.

A considerable number of memoirs, of unequal value, have been devoted to the theory of assemblages. Picard believes that some of these are without interest to mathematics and characterizes their theory as a sort of *metamathematics* from which paradoxes are not absent. Of the important work in the theory of assemblages one may select for mention that of Borel on measure, that of Lebesgue on integration, and that of Baire on the classification of functions.

Sometimes mathematical researches are presented with an excessive formalism and symbolism, incapable of leading to a new fact and of being utilized in other researches than in those in which they were created. But when these last conditions are not realized one can think that there is no real scientific progress. In this respect it seems that French mathematics has wisely restrained itself within proper bounds and has not created a chapter of science which affords merely an exercise in logic. Whenever our science has a tendency to become too formal let us turn again to the work of the great creators, such as Fourier, Cauchy and Poincaré, and learn from them anew how to create other chapters in that body of mathematics which is not a strange and mysterious science, but an essential part of the structure of natural philosophy.

R. D. CARMICHAEL.

SHORTER NOTICE.

Elementary Mathematical Analysis. By JOHN WESLEY YOUNG and FRANK MILLETT MORGAN. New York, The Macmillan Company, 1917. 548 pp.

THIS book is intended to meet the need which has been felt by many during the last years, of a more unified course in freshman mathematics which, without unduly neglecting matters of technique, places more than the usual emphasis upon a genuine understanding of the mathematical concepts and methods. The book has its faults, as have most books, but, in the opinion of the reviewer, it is a fine contribution to the problems in hand and is really a class room text.

The central theme is the idea of functionality. This idea is skillfully kept before the reader's mind, though in the consideration of some topics, as that of trigonometric relations, it is rather lost sight of. This seems to be inherent in the situation and it is possibly unwise and somewhat artificial to insist that everything in such a course must associate itself immediately with the idea of functionality, important and far reaching as that idea is.

The material of the book is presented under five general headings.

In Part I (Introductory Conceptions) thirty-two pages are devoted to the general subject of functions and their representation and thirty-one pages to algebraic principles and their connection with geometry. Great pains is taken to familiarize the reader with the notion of functionality and with the geometrical representation of functions. Many concrete examples are exhibited in detail. Under the second heading is given an elaborate discussion of numbers and their geometrical representation, a statement of the fundamental laws of algebra and a review of elementary algebraic technique.

In Part II two hundred and twenty-nine pages are devoted to the elementary functions: the linear function; the quadratic function; the cubic function; the function x^n ; the trigonometric functions with a special chapter on trigonometric relations; the logarithmic and exponential functions; numerical computation, including logarithmic solution of triangles, the slide rule, logarithmic paper; the implicit quadratic

functions. Under the last title equations of the second degree are classified according to their graphs. Thus the way is prepared for the conic section as the locus of a point which moves so that its distance from a fixed point is always equal to a constant times its distance from a fixed line.

Part III consists of 118 pages devoted to applications to geometry and contains a thorough treatment of the straight line, the circle, the conic sections, polar coordinates, and parametric equations.

In Part IV seventy-two pages are devoted to the consideration of such algebraic topics as permutations, combinations, probability, the binomial theorem, complex numbers, the general polynomial, the theory of equations and determinants.

Part V (forty-five pages) is a short treatment of the elements of solid analytic geometry.

The idea of slope as the limit of $\Delta y/\Delta x$ is introduced very early in Part II and is used wherever pertinent throughout the book. In addition to familiarizing the reader with this fundamental idea, the use of the slope facilitates the consideration of the various curves and enables the authors to make more of the subject of maxima and minima than is usual in a freshman course.

The trigonometric functions are defined at once for the general angle. The addition formulas are proved by means of the formulas for the rotation of axes, which have been previously considered. The chapter on logarithms is introduced by a careful, intuitive discussion of the exponential function, well calculated to give the reader a correct feeling as to the significance of exponents positive, negative, zero, rational, irrational.

The avowed purpose of the authors to place "more emphasis on insight and understanding of fundamental conceptions and modes of thought" is well carried out both in the text and in the problem lists. In the text are numerous questions tending to make the student read more thoughtfully and the problem lists contain many exercises of various degrees of difficulty which focus on the fundamental principles under consideration rather than on manipulation or technique. However there is a sufficient number of problems of all kinds and the lists are properly graded.

In the opinion of the reviewer, Part I is too long and (especially chapter two) contains too much discussion not well

chosen to introduce the subject to students during their first days in college. Chapter three, on the linear function, and certain parts of the sequel likewise seem to be somewhat verbose and indirect.

The book is well adapted for use by students who enter with trigonometry as well as by those who enter without trigonometry. The reviewer has felt that the program outlined in the preface is somewhat too ambitious and still believes that the average class will have some difficulty in covering the ground indicated. This emphasizes the need for greater brevity and directness in certain parts of the text. However by careful selection the needs of almost any class can be satisfied.

In the preface the authors announce the book as somewhat of an experiment and suggest a possible revision. It is too much to expect that, in these troublous times, such a revision may appear soon but it is to be hoped that it may not be delayed too long. In view of the experience of the authors themselves with the present book, a revised edition would surely be a finished product. The reviewer is, with the authors, in doubt as to the advisability of including more calculus in the text. Probably this can be determined only by experiment.

The book is attractive in print and in appearance and the figures are good. There are minor errors but the reviewer has made no effort to list these.

Any teacher who is interested in the matter of a more unified course for the freshman year would do well to give this book a careful examination. Any teacher of freshman mathematics who is interested in securing, on the part of his students, a better understanding and appreciation of the principles and methods of mathematics will find much here that is helpful whether or not he is interested in this particular type of course.

A. D. PITCHER.

NOTES.

THE concluding (October) number of volume 19 of the *Transactions of the American Mathematical Society* contains the following papers: "Spiral minimal surfaces," by J. K. WHITTEMORE; "On the group of isomorphisms of a certain extension of an abelian group," by L. C. MATHEWSON; "Concerning the zeros of the solutions of certain differential equations," by W. B. FITE; "Differentiation with respect to a function of limited variation," by P. J. DANIELL; "Linear integro-differential equations with a boundary condition," by M. T. HU; "On scalar and vector covariants of linear algebras," by OLIVE C. HAZLETT.

AT the meeting of the Edinburgh mathematical society on November 8, the following paper was read: By J. E. A. STEGGAL, "Notes on the homographic transformation of the cubic and biquadratic."

THERE has recently been issued by the Bureau of Education at Washington a Bulletin on the training of teachers of mathematics for secondary schools of the countries represented in the International commission on the teaching of mathematics. This Bulletin has been prepared by Professor R. C. ARCHIBALD, of Brown University. It is a work of nearly three hundred pages, giving in great detail the requirements set by the various governments for a teacher of secondary mathematics. The Bureau of Education has a limited number of copies for free distribution to those who are particularly interested in the work, and copies can be obtained from the Superintendent of Documents, Government Printing Office at Washington, D. C., at 30 cents each.

HARVARD UNIVERSITY. The winter term of the graduate school of arts and sciences will begin January 2 and close March 20. The spring term will begin March 28 and close with Commencement Day, June 19. It is expected that a summer term of eleven weeks will be arranged, so that a student beginning his work in the winter may complete a full year of residence by September. Those to whom fellowships or scholarships

had been assigned for the academic year 1918-19, but who, because of national service, resigned these, should apply promptly for reappointment. A certain number of fellowships and scholarships are now vacant, for which applications will be received.

PROFESSOR C. CARATHÉODORY, of the University of Göttingen, has been appointed professor of mathematics at the University of Berlin, as successor to the late Professor G. FROBENIUS.

PROFESSOR G. C. EVANS, of the Rice Institute, has been commissioned captain of ordnance in the United States Army and sent to France on a special mission.

LIEUTENANT T. R. HOLLCROFT has been released from the teaching staff of the school of artillery at Camp Taylor, and has resumed the duties of his professorship of mathematics at Wells College. During his absence the position was filled by Mrs. Hollcroft.

DR. T. M. SIMPSON, of the University of Wisconsin, has been made professor and head of the department of mathematics at the University of Florida, as successor to the late Professor H. G. KEPPEL.

PROFESSOR J. W. YOUNG, of Dartmouth College, has been made director of the instruction in mathematics given under the National war work council of the Young Men's Christian Association. During his absence and that of Professors HASKINS and BILL, also engaged in war work, the administration of the department of mathematics is in the hands of Professor F. M. MORGAN.

DR. S. D. ZELDIN, of the College of Hawaii, has been appointed professor of mathematics at Olivet College.

PROFESSOR W. O. MENDENHALL, of Earlham College, has been elected to the presidency of the Friends University, Wichita, Kansas.

MR. C. A. HUTCHINSON has been appointed instructor in mathematics at the University of Colorado.

DR. J. E. McATEE, of the University of Illinois, died December 2, 1918.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BELL (R. J. T.). See FROST (P.).

BENEDICT (H. Y.). See KARPINSKI (L. C.).

CALHOUN (J. W.). See KARPINSKI (L. C.).

COTES (R.). See KOWALEWSKI (A.).

CULLIS (C. E.). *Matrices and determinoids. Volume 2.* Cambridge University Press, 1918. 24+555 pp. 42 s.

DONADT (A.). See LÜBSEN (H. B.).

FISCHER (L. A.). See HUNTINGTON (E. V.).

FROST (P.). *An elementary treatise on curve tracing.* 4th edition, by R. J. T. Bell. London, Macmillan, 1918. 16+210 pp. 12s. 6d.

GAUSS (C. F.). See KOWALEWSKI (A.).

GIGNANO (E.). *Essays in scientific synthesis.* Translated by W. J. Greenstreet. Chicago, Open Court, 1918. 235 pp. \$2.00

GREENSTREET (W. J.). See GIGNANO (E.).

JACOBI (C. G. J.). See KOWALEWSKI (A.).

KARPINSKI (L. C.), BENEDICT (H. Y.) and CALHOUN (J. W.). *Unified mathematics.* Boston, Heath, 1918. 8 + 522 pp. \$2.80

KOWALEWSKI (A.). *Newton, Cotes, Gauss, Jacobi. Vier grundlegende Abhandlungen über Interpolation und genäherte Quadratur.* Herausgegeben von A. Kowalewski. Leipzig, Veit, 1917. Geh. 104 pp. M. 4.80

LÜBSEN (H. B.). *Einleitung in die Infinitesimalrechnung (Differential- und Integralrechnung) zum Selbstunterricht.* Mit Rücksicht auf das Notwendigste und Wichtigste. Neu bearbeitet von A. Donadt. 9te Auflage. Leipzig, F. Brandstetter, 1916. 8vo. 7+440 pp. M. 8.00.

NEWTON (I.). See KOWALEWSKI (A.).

PIRA (K.). *Om idesystem och talsystem (Smärre Skrifter utg. af Boströmsförbundet, 38-40).* Del 1-3. Stockholm, A. B. A. Johansen, 1913-14. 8vo. 126 + 108 + 184 pp. Kr.1.50+1.25+2.00

RANSOM (W. R.). *Freshman mathematics brief, simple, practical for the Freshman college classes.* New York, Longmans, 1918. 320 pp. \$1.50

TOLEDO (L. O. de). *Elementos de aritmética universal. Tomo 2: Coordinatoria, determinantes, algoritmos ilimitados.* Madrid, Librería de la Vda é Hijos de Murillo, 1916. 8vo. 421 pp. Pes. 14.00

VOIGT (A.). Die Teilbarkeit der Potenzsummen und die Lösung des Fermat'schen Problems. Eine zahlentheoretische Untersuchung. Frankfurt a.M., M. Diesterweg, 1916. 8vo. 12+105 pp. M. 2:00

II. ELEMENTARY MATHEMATICS.

ALZÁA (F. de). Nociones de geometría plana. Buenos Aires, A. Garcia Santos, 1917.

GOMEZ RUIZ (A.). Tratado de trigonometria plana, circular y hiperbolica. (Memorias de la Real Academia de Ciencias, Madrid.) Madrid, De Fortanet, 1917. 4to. 318 pp.

KING (W. J.). A condensed geometry review with answers. Annapolis, Md., C. G. Feldmeyer, 1918. 8vo. 35 pp.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe A: Für Gymnasien. 2ter Teil: Die Oberstufe. 4te, umgearbeitete Auflage. Leipzig, Teubner, 1917. 14+360 pp.

TOLEDO (L. O. de). Tratado de álgebra. Tomo 1: Parte elemental. Seg. edition. Madrid, Libreria General de Victoriano Suárez, 1914. 8vo. 11+362 pp. Pes. 10.00

— Tratado de trigonometria rectilínea y esférica. Seg. edition. Madrid, Libreria General de Victoriano Suárez, 1914. 8vo. 285 pp. Pes. 8.00

III. APPLIED MATHEMATICS.

ARRHENIUS (S.). The destinies of the stars, by S. Arrhenius. Authorized translation from the Swedish by J. E. Fries. New York, Putnam, 1918. \$1.50

ATTWOOD (E. L.). Textbook of theoretical naval architecture. 9th impression. New York, Longmans, 1918. 8vo. 12+494 pp. \$3.50

BECC (A.). Applications industrielles de l'électricité. Transmission, distribution, et utilisation de l'énergie électrique. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 231 pp. Fr. 12.00

DEVILLERS (R.). Le moteur à explosions. Préface de C. Faroux. Paris, Dunod et Pinat, 1917. 8vo. 8+422 pp. Fr. 8.25

DONNER (A.). See FESTSKRIFT.

DORGEOT (E.). La mécanique appliquée, théorique, numérique et graphique. Paris, Dunod et Pinat, 1918. 4to. 613 pp. Fr. 39.00

FAROUX (C.). See DEVILLERS (R.).

FESTSKRIFT tillegnad Anders Donner på hans sextioårsdag den 5 nov. 1914 af forne elever. Helsingfors, J. Simellii Arfvingars Boktryckeria-Ktiebolog, 1915. 4to. 410 pp.

FRIES (J. E.). See ARRHENIUS (S.).

FÜRST (A.) UND MOSZKOWSKI (A.). Das Buch der 1000 Wunder. München, A. Lange, 1917. Geh. M. 6.00

HOLMS (A. C.). Practical shipbuilding. A treatise on the structural design and building of modern steam vessels. Volume 1: Text. Volume 2: Diagrams and illustrations. 6th impression. New York, Longmans, 1918. \$20.00

HUNTINGTON (E. V.). Handbook of mathematics for engineers by E. V. Huntington with tables of weights and measures by L. A. Fischer. New York, McGraw-Hill, 1918. 8vo. 191 pp. \$1.50

HUTCHINSON (R. W.). Advanced textbook of magnetism and electricity. London, University Tutorial Press, 1917.

KAYE (G. W. C.). Tables of physical and chemical constants and some mathematical functions. 3d edition. New York, Longmans, 1918. 8vo. 8+153 pp. \$2.25

— X-rays. London, Longmans, 1917.

KAUFMANN (A.). Theorie und Methoden der Statistik. Ein Lehr- und Lesebuch für Studierende und Praktiker. Tübingen, Mohr, 1913. 8vo. 12+540 pp.

LOW (D. A.). A pocketbook for mechanical engineers. 7th impression. New York, Longmans, 1918. 8vo. 8+740 pp. \$3.00

MAUNDER (E. W.). The stars and how to identify them. London, Kelly, 1918. 63 pp.

MOSZKOWSKI (A.). See FÜRST (A.).

PLUMMER (H. C.). An introductory treatise on dynamical astronomy. Cambridge, University Press, 1918. Royal 8vo. 18s.

SCHOUTEN (W. J. A.). On the determination of the principal laws of statistical astronomy. Amsterdam, W. Kirchner, 1918. 128 pp.

SMITH (H. E.). Mechanics; prepared for the use of midshipmen and adopted as the textbook at the Naval Academy. Annapolis, U. S. Naval Institute, 1918. 8vo. 11+269 pp. \$3.60

STEINMETZ (C. P.). Engineering mathematics; a series of lectures delivered at Union College. 3d edition, revised and enlarged. New York, McGraw-Hill, 1917. 8vo. 49+321 pp. \$3.00

STORMER (C.). Sur un problème relatif au mouvement des corpuscules électriques dans l'espace cosmique. (Utgitt for Fridtjof Nansens Fond.) Kristiania, J. Dybwad, 1916. 8vo. 60+91 pp.

SVENSEN (C. L.). Essentials of drafting; text and problem book for apprentice, trade and evening technical schools. New York, Van Nostrand, 1918. 8vo. 7-14+184+31 pp. \$1.50

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM LECTURES, VOLUME V

THE CAMBRIDGE COLLOQUIUM
1916

PART I
FUNCTIONALS AND THEIR APPLICATIONS
SELECTED TOPICS, INCLUDING
INTEGRAL EQUATIONS

BY
GRIFFITH CONRAD EVANS

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THE DECEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-second regular meeting of the San Francisco Section, postponed from October 26, was held at the University of California on Saturday, December 14, 1918. The total attendance was thirteen, including the following nine members of the Society:

Professor B. A. Bernstein, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Dr. F. R. Morris, Professor C. A. Noble, and Dr. Pauline Sperry.

The chairman, Professor Manning, opened the meeting, later giving way to Professor Cajori, the newly elected chairman. At the election of officers the following were chosen for one year: Chairman, Professor Florian Cajori; Secretary, Professor B. A. Bernstein; Programme committee, Professor L. M. Hoskins, Professor W. A. Manning, and Professor B. A. Bernstein.

It was decided to hold the next fall meeting on Saturday, October 25, 1919, at the University of California.

The following papers were read:

(1) Professor FLORIAN CAJORI: "A note on the history of Playfair's axiom."

(2) Professor FLORIAN CAJORI: "On the Aristotelian tract, *De lineis insecabilibus*."

(3) Professor M. W. HASKELL: "Triangles inscribed and circumscribed to a plane cubic."

(4) Professor D. N. LEHMER: "A non-tentative method of solving the indeterminate equation $Ax + By + Cz + \dots = n$."

(5) Professor E. T. BELL: "A partial isomorph of trigonometry."

(6) Professor E. T. BELL: "Arithmetical paraphrases (I)."

(7) Professor E. T. BELL: "Arithmetical paraphrases (II); elliptic and theta series."

(8) Professor E. T. BELL: "Arithmetical paraphrases (III); chiefly for the doubly periodic functions of the first, second, and third kinds."

(9) Professor E. T. BELL: "Arithmetical paraphrases (IV); class number formulas."

Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. Professor Cajori points out that Playfair's axiom on parallel lines, now usually attributed to William Ludlam (1785), was used by Joseph Fenn in Dublin, in 1769.

2. The *De lineis insecabilibus* enumerates 5 arguments, current in Aristotle's day, in favor of the existence of indivisible lines, 22 considerations showing the invalidity of those arguments, and 26 considerations tending to disprove the view that a line is made up of points. Some of the arguments are rigorous. In Professor Cajori's opinion the tract deserves a place in the history of mathematics.

3. Professor Haskell shows that there are twenty-four triangles inscribed and circumscribed to a non-singular cubic. These twenty-four triangles occur in four sets of six each, corresponding to the four inflexional triangles, so that in each set three are triply in perspective with the other three, the centers of perspective for each pair being three collinear inflexions. The vertices of each pair lie on a conic.

For a cubic with a double point, the total number of triangles is two, which are real if the double point is isolated and imaginary otherwise. A cuspidal cubic has no such triangle.

Further, with respect to each inflexional line there is a six-point involution on the cubic, determined by a pencil of conics, where each set of six points are the vertices of two triangles triply in perspective with respect to the inflexions lying on the given line. These two triangles become coincident in three cases, the vertices then being sextactic points.

4. The solution of the indeterminate equation $Ax + By = n$ by means of the regular continued fraction representing A/B furnishes an example of the very few non-tentative methods available in the theory of numbers. Professor Lehmer uses the theory of continued fractions of higher orders to obtain all the solutions of the equation $Ax + By + Cz + \dots = n$.

The paper will be offered to the *American Journal* as part of a memoir on continued fractions of higher orders, an abstract of which has already been inserted in the *Proceedings of the National Academy of Sciences*.

5. Professor Bell defines a function to be even or odd in a set of variables according as it does not or does change sign when the signs of all the variables are changed simultaneously. Let X_i, Y_j denote sets containing a_i, b_j variables respectively; then

$$f(X_1, X_2, \dots, X_r | Y_1, \dots, Y_s)$$

is defined to be even in each X , odd in each Y , and its parity is the symbol

$$p(a_1, a_2, \dots, a_r | b_1, b_2, \dots, b_s)$$

whose significance is obvious. If

$$a_i = 1 = b_j, (i = 1, \dots, r; j = 1, \dots, s),$$

the parity is written

$$p(1^r | 1^s).$$

It is shown that any function whose parity is

$$p(a_1, a_2, \dots, a_r | b_1, b_2, \dots, b_s)$$

is linearly expressible in terms of 2^κ properly chosen functions F , whose parities are all of the form

$$p(1^\alpha | 1^\beta),$$

where

$$\alpha + \beta = \kappa,$$

and

$$\kappa = (a_1 + a_2 + \dots + a_r) + (b_1 + b_2 + \dots + b_s) - (r + s).$$

An arbitrary function (one neither odd nor even in any set of its variables) of k variables is similarly expressible, the number of the F being 2^k . These results are useful in the arithmetical applications of the elliptic and theta functions.

6. From 1858 to 1865 Liouville published intermittently a series of eighteen memoirs, "Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres," wherein he gave without proof many remarkable theorems which he and others applied to various questions of interest, including class number formulas. Proofs by extensions of the method used by Dirichlet in proving Jacobi's four square theorem have been given for most of Liouville's general results by H. J. S. Smith (1865), T. Pepin (1889-90), G. B. Mathews

(1892), and E. Meissner (1907). But, as remarked by Bachmann (1910): "Eine zusammenhängende systematische Herleitung und Verbindung dieser Formeln (Liouville's), welche ihren Quell und die Prinzipien klarlegte, nach denen die in den Formeln auftretenden Argumente der Funktionen, auf welche sie sich beziehen, zu wählen sind, würde sehr wertvoll sein." By means of the theorems in his first paper and another simple principle, Professor Bell shows that the classical expansions in the theory of elliptic and theta functions may be paraphrased directly into theorems of the Liouville kind; moreover that a simple method is thereby provided for the discovery at will of new results of the same general sort. Liouville's formulas are nearly all paraphrases of Jacobi's expansions in the *Fundamenta Nova* and in the memoirs on rotation.

7. In his second paper on paraphrases Professor Bell prepares and collects elliptic and theta expansions in a form suitable for paraphrase, and gives sixteen trigonometric identities basic for Liouville's numerous (unproved) theorems on the quadratic forms of certain primes. The series are classified according to the forms of the divisors d, δ appearing in the sine, cosine coefficients of the powers of q . Similar series for sets of non-doubly periodic theta quotients, such as Biehler's, Hermite's, and Humbert's, are also given.

8. In his third paper on paraphrases, Professor Bell applies the methods of his first to a few of the series in the second. Among other results it is shown that all the formulas in Liouville's fifth memoir (and many more) are immediate paraphrases of a few classic formulas relating to the doubly periodic functions of the second kind. Illustrative of the simplicity of the method, the formulas of Liouville's sixth memoir follow at once from Jacobi's series for $\text{sn}^3 u$.

9. Professor Bell's fourth paper on paraphrases applies the method to the series for non-doubly periodic theta quotients given by Hermite (1862) and extended by G. Humbert (1907). The resulting paraphrases involve class numbers and arbitrary functions odd or even in one or more variables. It is shown how such paraphrases are related to the number of representations of an integer as a sum of 3, 5, 7, 9, 11, ... squares.

B. A. BERNSTEIN,
Secretary of the Section.

THE SCIENTIFIC WORK OF MAXIME BÔCHER.

BY PROFESSOR GEORGE D. BIRKHOFF.

WITH the recent death of Professor Maxime Bôcher at only fifty-one years of age American mathematics has suffered a heavy loss. Our task in the following pages is to review and appreciate his notable mathematical work.*

His researches cluster about Laplace's equation $\Delta u = 0$, which is the very heart of modern analysis. Here one stands in natural contact with mathematical physics, the theory of linear differential equations both total and partial, the theory of functions of a complex variable, and thus directly or indirectly with a great part of mathematics.

His interest in the field of potential theory began in undergraduate days at Harvard University through courses given by Professors Byerly and B. O. Peirce. There is still on file at the Harvard library an undergraduate honor thesis entitled "A thesis on three systems of parabolic coördinates," written by him in 1888. Under the circumstances it was inevitable that he should use formal methods in dealing with his topic, but a purpose to penetrate further is found in the concluding sentences. No better opportunity for fulfilling such a purpose could have been granted than was given by his graduate work under Felix Klein at Göttingen (1888-1891).

In the lectures on Lamé's functions which Klein delivered in the winter of 1889-1890 his point of departure was the cyclidic coördinate system of Darboux. This system of coördinates was known to be so general as to include nearly all of the many types of coördinates useful in potential theory, and Wangerin had shown (1875-1876) how solutions of Laplace's equation existed in the form of triple products, each factor being a function of one of the three cyclidic coördinates. After presenting this earlier work Klein extended his "oscillation theorem" for the case of elliptic coördinates (1881) to the more general cyclidic coördinates. By this means he was able to attack the problem of setting up a potential function taking on given values over the surface of a solid bounded by

* An account of his life and service by Professor Osgood will appear in a later number of the BULLETIN.

six or fewer confocal cyclides. This function was given by a series of the triple "Lamé's" products discovered by Wangerin.

Klein also aimed to get at the various forms of series and integrals previously employed in potential theory as actual limiting cases, and thus to bring out the underlying unity in an extensive field of mathematics.

The task which Bôcher undertook was to carry through the program sketched by Klein. He did this admirably in his first mathematical paper "Ueber die Reihenentwickelungen der Potentialtheorie," which appeared in 1891 and which served both as a prize essay and as his doctor's dissertation at Göttingen.* But the space available was so brief that he was only able to outline results without giving their proofs.

One must look to his book with the same title,† published three years later, for an adequate treatment of the subject. Here is also to be found original work not outlined in his dissertation. It was characteristic that he did not call attention explicitly to the new advances although these formed his most important scientific work in the years 1891-1894. We turn now to a consideration of this book, which thus contains nearly all that he did before 1895.

Besides giving the classification of all types of confocal cyclides in the real domain and of the corresponding Lamé's products, as sketched by Klein, Bôcher determined to what extent the theorem of oscillation holds in the degenerate cases and found an interesting variety of possibilities.

The difficulties presented by these degenerate cases are decidedly greater than those of the general case when the singular points e_i ($i = 1, 2, 3, 4, 5$) of the Lamé's linear differential equation are regular with exponents $0, 1/2$. A very simple degenerate case is that arising when two such points coincide in a single point and one of the two intervals $(m_1, m_2), (n_1, n_2)$ under consideration ends at this point. By an extension of Klein's geometric method, he proved that the theorem of oscillation fails to hold even here.

More specifically, the facts are as follows. In the general case the oscillation theorem states that for any choice of integers m, n ($m, n \geq 0$) there is a unique choice of the two ac-

* This paper appears as (2) in the chronological list of papers given at the end of the present article. Hereafter footnote references to papers will be made by number.

† (15).

cessory parameters in the differential equation, yielding solutions u_1, u_2 such that u_1 vanishes at m_1 and m_2 , and m times for $m_1 < x < m_2$, while u_2 vanishes at n_1 and n_2 , and n times for $n_1 < x < n_2$. If now, for instance, m_1 lies at the double singular point $e_1 = e_2$, while $m_1 < m_2 < e_3 < e_4 < n_1 < n_2 < e_5$, there exist such solutions u_1, u_2 only if $n > r_m$ where r_m is an integer increasing indefinitely with m . But, to compensate for this deficiency of solutions of the boundary value problem, Bôcher found it necessary to introduce solutions u_{1k}, u_{2k} dependent on n and a continuous real parameter k such that u_1 vanishes at m_2 and infinitely often for $m_1 < x < m_2$ although remaining finite, while u_2 vanishes at n_1 and n_2 , and n times for $n_1 < x < n_2$.

The corresponding expansion in Lamé's products presents a remarkable form under these circumstances, for it is made up of a series and an integral component. In another case this type of expansion takes the form of an integral augmented by a finite number of complementary terms, as he had pointed out in an important paper "On some applications of Bessel's functions with pure imaginary index,"* published in 1892 in the *Annals of Mathematics*.

Although dealing satisfactorily with the oscillation theorem in the case specified above and other similar cases, Bôcher did not discuss adequately the case in which three or more singular points unite to form an irregular singular point.† Indeed it appears that he fell into an error of reasoning as follows. If the irregular point be taken at $t = +\infty$ the Lamé's equation has the form

$$\frac{d^2y}{dt^2} = \varphi y,$$

where in the case under consideration φ has a limit $\varphi_0 \neq 0$ as t becomes infinite. The lemma which Bôcher then sought to prove‡ was that there always exists a solution y finite for $t \geq T$ and not identically zero. His proof for the case $\varphi_0 > 0$ is essentially correct. Here he interpreted the equation above as the equation of motion of a particle distant y from a point O of its line of motion and repelled from it with a force φy .

* (7). In passing, attention may also be called to a slightly earlier article (3) on Bessel's functions.

† See (15), p. 179.

‡ See (15), p. 177.

The gist of the argument employed is that one can find an initial velocity of projection toward O just sufficient to carry it into that point as a limiting position. This part of the lemma constitutes a very simple and interesting theorem concerning a special type of irregular point. In the case $\varphi_0 < 0$, however, using a similar dynamical interpretation, he argued* "we have infinitely many oscillations as we approach $t = +\infty$, and since the attractive force is not infinitely weak, the amplitudes of the oscillations remain finite." This argument appears insufficient although the lemma as stated for Lamé's equation is probably correct.† To satisfactorily complete the discussion it would seem to be necessary to call in the explicit analytic theory of the irregular singular point, since the corresponding theory of the regular singular point is required in the simpler cases.‡

In his book Bôcher considered the boundary problem under *periodic* conditions, when the interval between two adjacent singular points is taken an even number of times and is regarded as closed; this case arises, for example, when the solid in the potential problem is a complete ellipsoid. Here the function φ in the linear differential equation above written is an even doubly periodic function with real period. By the aid of these properties of φ he reduced the new boundary problem to one of the ordinary type.

Likewise in treating the roots of Lamé's polynomials he made a distinct advance by extending the dynamical method of Stieltjes from the real axis to the complex plane. Thus he was able to prove that the roots of these polynomials lie within the triangle whose vertices are the three finite singular points of the corresponding Lamé's equation.

Finally we may note that at the end of his book he obtained all Lamé's products satisfying the equation $\Delta u + k^2 u = 0$.

The determinative effect of the dissertation and book upon the direction of Bôcher's later researches was very great. In the first place he had used sphere geometry and the algebra of elementary divisors as essential tools in analysis; his resulting interest in the fundamental parts of geometry and algebra never subsided, and some of his research lies in these fields.

* (15), p. 178. The translation is not literal.

† In this connection see (7), p. 150, footnotes.

‡ Since the above was written Professor Osgood has disposed of the question at issue by elementary means. See his note in this number of the BULLETIN.

But, more important still, he was brought into contact with open mathematical questions. The most vital of these questions from the purely mathematical point of view was doubtless the very difficult analytical question of convergence and representation presented by the series of Lamé's products. This was the outstanding problem which Klein emphasized,* but to which Bôcher seems never to have given particular attention. Another more practical direction of effort was afforded by the task of giving rigorous and accessible form to the work of Sturm and Klein on the real solutions of ordinary linear differential equations and then going on further in this overlooked but attractive field of research. It was primarily to this task that he now turned.

In 1897 he published an article in the BULLETIN† showing the immediate usefulness of Sturm's theorems for fixing the distribution of the roots of Bessel's functions with real index. A year later in the same place he presented the fundamentals of Sturm's work in simplified rigorous form, and gave the first analytic proof of Klein's theorem of oscillation.‡

Reference should also be made to his article on the boundary problems of ordinary differential equations which appeared in the German mathematical encyclopedia in 1900. This article together with his address on "Boundary problems in one dimension" before the Fifth International Congress of Mathematicians in 1912 give an excellent account of this field to the latter date.

Bôcher wrote a considerable number of other papers in this same field.§ Perhaps the most important of these are the three to which we will refer first and which appeared in the beginning volumes of the *Transactions*.

His paper "Application of a method of d'Alembert to the proof of Sturm's theorems of comparison" (1900) contained an elegant proof of what Bôcher had called the theorems of comparison. His method was entirely different from Sturm's, being based on the Riccati's resolvent equation, and was very simple.

In the second of these papers "On certain pairs of trans-

* See the concluding pages of his 1889-1890 lectures on Lamé's functions and his preface to Bôcher's book (15).

† (26). See (38) also.

‡ (30), (31), (35).

§ (32), (38), (42), (46), (48), (49), (50), (55), (65), (81), (85), (92), (93), (100).

cidental functions whose roots separate each other"* (1901) his starting point was the linear differential equation

$$y'' + py' + qy = 0,$$

and a pair of linear forms in y, y' ,

$$\Phi = \varphi_2 y' - \varphi_1 y, \quad \Psi = \psi_2 y' - \psi_1 y.$$

These latter satisfy a "homogeneous Riccati's equation"

$$(\varphi_1 \psi_2 - \varphi_2 \psi_1)(\Phi' \Psi - \Phi \Psi') + A\Phi^2 + B\Phi\Psi + C\Psi^2 = 0,$$

and Bôcher considered the relation of the roots of Φ, Ψ .

He notes first that Φ, Ψ cannot vanish together unless $\varphi_1 \psi_2 - \varphi_2 \psi_1 = 0$, for otherwise $y = y' = 0$. In order that Φ, Ψ cannot vanish together it is thus sufficient to assume $\varphi_1 \psi_2 - \varphi_2 \psi_1 \neq 0$. Also if $\Phi = 0$, then $\Phi' \neq 0$ if $C \neq 0$, by the above equation. A like remark holds for Ψ . Hence the roots of Φ, Ψ are simple if $A \neq 0, C \neq 0$.

Under these hypotheses between any pair of adjacent roots of Φ there must be a root of Ψ . For if Ψ has no such root the homogeneous Riccati's equation at these roots shows that Φ' has one and the same sign at both roots, which is impossible. Likewise between any pair of adjacent roots of Ψ there must be a root of Φ .

Hence *the roots of Φ, Ψ separate each other if*

$$\varphi_1 \psi_2 - \varphi_2 \psi_1 \neq 0, \quad A \neq 0, \quad C \neq 0.$$

This is the third theorem of the paper. The sixth theorem gives similar conditions sufficient to ensure cyclical separation of the roots of three linear forms.

Here Bôcher not only achieved greater generality and simplicity than Sturm but, as I wish to point out, he has reached a maximum of generality.

For, let y_1, y_2 be any pair of linearly independent solutions yielding the values Φ_1, Φ_2 and Ψ_1, Ψ_2 of Φ and Ψ . Then

$$\Phi = c_1 \Phi_1 + c_2 \Phi_2, \quad \Psi = c_1 \Psi_1 + c_2 \Psi_2$$

are the general values of Φ, Ψ . If Φ_1, Φ_2 are regarded as the homogeneous coordinates of a point P in the projective line, Φ vanishes if P coincides with $E \equiv (-c_2, c_1)$; similarly Ψ vanishes if $Q = (\Psi_1, \Psi_2)$ coincides with the same point E .

* See also (100).

Clearly the roots of Φ, Ψ will only be distinct for all values of c_1, c_2 if $\varphi_1\psi_2 - \varphi_2\psi_1 \neq 0$. Moreover, if these roots are to separate each other for all values of c_1, c_2 , the points P, Q must pass *any point* E in alternation. This is only possible if P, Q never reverse their direction of motion; in other words the Wronskians of Φ_1, Φ_2 and of Ψ_1, Ψ_2 must be of invariant signs. Taking into account the fact that $y_1y_2' - y_1'y_2$ is not zero, this gives precisely the conditions $A \neq 0, C \neq 0$.

This same geometric interpretation shows a similar generality in the other theorems.

Of like completeness is the third paper "On the real solutions of systems of two homogeneous linear differential equations of the first order" (1902), where he treated analogous questions and also derived comparison theorems.

It was a matter of primary interest with him to vary proofs of known theorems as well as to discover new theorems. An illustration in point is afforded by his treatment of the elementary separation theorem for the roots of linearly independent solutions y_1, y_2 of an ordinary linear differential equation of the second order.

Here he first gave a very brief proof* based on the function y_1/y_2 : if y_1 vanishes at a and b but not for $a < x < b$, while y_2 is not zero for $a \leq x \leq b$, then the derivative of y_1/y_2 is of one sign for $a < x < b$ since $y_1y_2' - y_1'y_2 \neq 0$. This is impossible. By this argument and a like argument based on y_2/y_1 it follows that the roots of y_1, y_2 separate each other. In the same place† he isolates a geometric proof implicitly given by Klein depending on the fact that if y_1, y_2 be taken as homogeneous coördinates of a point in the projective line then $y_1y_2' - y_1'y_2 \neq 0$ is the condition that this point moves continually in one sense. Later he gave a second analytic proof based on the function

$$\frac{y_1'}{y_1} - \frac{y_2'}{y_2}, \ddagger$$

and also a second geometric proof* based on the vector $y_1 + \sqrt{-1}y_2$ in the complex plane which will rotate continually in one sense if $y_1y_2' - y_2y_1' \neq 0$.

* (26), p. 210.

† Footnote, p. 210.

‡ (48).

§ (99), pp. 46-47.

It was not easy for him to believe that the methods of Sturm were inadequate to deal with any particular boundary problem in one dimension. The problem for periodic conditions, which had been formulated by him in his encyclopedia article, was first successfully attacked by Mason in 1903–1904 by means of the calculus of variations. In a very interesting note published in 1905,* Bôcher showed that the principal result fell out immediately by the methods of Sturm, and that these methods were applicable under much more general conditions. Likewise in his address before the Fifth International Congress of Mathematicians alluded to above he noted that the equation

$$\frac{d}{dx} \left(k \frac{du}{dx} \right) + (\lambda g - l)u = 0, \quad l < 0,$$

(λ a parameter) comes directly under the case treated by Sturm after division by $|\lambda|$ even if g changes sign. This simple remark disposed of the necessity of treating this case separately, as had been done earlier.

Bôcher was interested in all phases of the theory of ordinary linear differential equations with real independent variable. Having seen the gap in the theory of the regular singular point for real independent variable when the coefficients are not analytic, he proved that theorems analogous to those given by Fuchs in the complex domain are true.† It was necessary here to replace the power series treatment by a variation of the method of successive approximation which has been seen later to afford a new approach to the theory of the regular singular point in the complex domain.

He also did some work in the field of fundamental existence theorems for linear differential equations.‡ He showed that it is sufficient to impose the condition of integrability (joined with other conditions) upon the coefficients in place of Peano's condition of continuity,§ and thus advanced beyond Peano. Bôcher seems also to have been the first to prove that the solutions of a linear differential system are continuous functionals of the coefficients.||

* (65).

† (37), (40), (41).

‡ (32), (37), (56).

§ (56), p. 311.

|| (56), p. 315; (55), p. 208.

In 1901 he published a paper on "Green's functions in space of one dimension," in which he pointed out that the Green's function for the equation of Laplace in one dimension $y'' = 0$, exhibited by Burkhardt in 1894, might be extended to the general n th order ordinary linear differential equation with fairly general boundary conditions. These extended Green's functions have turned out to be of great importance. Later he returned to the subject of Green's functions with the most general linear boundary conditions and set up these functions for linear difference equations.* Also he extended the notion of adjoint boundary conditions to very general cases.†

We have now referred briefly to the most important of his researches on ordinary linear differential equations with *real* independent variable. In this domain his best work is perhaps to be found. Directly springing from this field were his researches on linear dependence of functions of a single real variable‡—an important topic which he was the first to isolate sufficiently from the field of linear differential equations.

His paper on "The roots of polynomials which satisfy certain linear differential equations of the second order"§ lies in the field of ordinary linear differential equations with a *complex* variable. Here he generalizes further the extension of the method of Stieltjes which he had employed in dealing with Lamé's polynomials.

The series arising in mathematical physics had been Bôcher's point of departure. Indeed it is the existence of these series which constitutes the main importance of the boundary value problems of linear differential equations. Nevertheless he gave special attention only to Fourier's series which he took up in an expository article in the *Annals of Mathematics* for 1906.|| Here he called attention to the remarkable phenomenon exhibited by a Fourier's series near a point of discontinuity, previously noted by Gibbs and called "Gibbs's phenomenon" by Bôcher who gave the first adequate treatment of it.¶

His contributions to the theory of the harmonic function in two dimensions are elegant and distinctly important.

* (81).

† (85).

‡ (43), (45), (47), (51), (97).

§ (29).

|| (67). See also (89).

¶ Reference may also be made here to the short note on infinite series (60).

The first of these occurs incidentally in his paper "Gauss's third proof of the fundamental theorem of algebra."* It consists in a proof of the average value theorem by means of Gauss's theorem for the circle, which in polar coordinates r, φ is

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\varphi = 0.$$

Integrating with respect to r from 0 to a and reversing the order of integration, we get

$$\int_0^{2\pi} (u(a, \varphi) - u(0, \varphi)) d\varphi = 0,$$

whence the average value theorem follows at once. This very neat proof was probably suggested by the artifice used by Gauss in his third proof of the fundamental theorem of algebra.

The "Note on Poisson's integral" (1898) gives a more natural interpretation of Poisson's integral than had been stated before. By the average value theorem a harmonic function is the average of its values on any circle with its center at the given point. He generalized this theorem in the spirit of the geometry of inversion and thus reached a visual interpretation of Poisson's integral which may be formulated as follows: The value of a harmonic function at any point within a circle is the average of its values as read by an observer at the point who turns with uniform angular velocity, if the rays of light to his eye take the form of circular arcs orthogonal to the given circle.

According to Riemann's program, the theory of harmonic functions requires a development independent of the theory of functions of a complex variable. In 1905 Bôcher demonstrated† that a harmonic function could not become infinite at a point unless it was of the form $C \log r + v$, where C is a constant, r is the distance from a variable point to the given point and v is harmonic at that point. This theorem corresponds to the fundamental theorem in functions of a complex variable which states that if $f(z)$ becomes infinite at the isolated singular point $z = a$, then $f(z)$ is of the form $(z - a)^{-r}g(z)$ where r is a positive integer and $g(z)$ is analytic and not zero

* (17), p. 206.

† (59).

at $z = a$. He demonstrated further that a similar theorem holds for large classes of linear partial differential equations.

Another extremely interesting paper "On harmonic functions in two dimensions" appeared in 1906. Here he defines u to be harmonic if it is single valued and continuous with continuous first partial derivatives and satisfies Gauss's theorem for every circle. If u possessed continuous second partial derivatives also it would then follow at once by Green's theorem that u is harmonic in the customary sense. But it is the merit of Bôcher's paper to have proved that u is harmonic in the ordinary sense without further assumptions. On the basis of the definition made, the average value theorem is first deduced as outlined above. Also if s', n' are the new variables s, n after an inversion (taking circles into circles) we have

$$0 = \int \frac{\partial u}{\partial n} ds = \int \frac{\partial u}{\partial n'} ds'$$

along corresponding circles, since $ds'/dn' = ds/dn$ (the inversion being conformal). Thus u is "harmonic" in the transformed plane also, so that the definition is invariant under inversion. Hence Poisson's integral formula, which comes from the average value theorem by inversion, also holds, and u is harmonic in the ordinary sense.

He also determined the precise region of convergence of the real power series in x, y for any harmonic function $u(x, y)$.*

In connection with his papers on harmonic functions in two dimensions it is natural to call to mind his early paper "On the differential equation $\Delta u + k^2 u = 0$ " (1893), which is taken in two dimensions. The " u -functions" so defined give a generalization of harmonic functions which he treated by means of the fact that $u(x, y)e^{kz}$ satisfies Laplace's equation in three dimensions. A similar method had been employed earlier by Klein.

Practically none of Bôcher's work lies directly in the field of functions of a complex variable.†

We have still to consider his contributions in the fields of algebra and geometry. In the early paper on the fundamental theorem of algebra cited above he made clear how, by taking for granted a few theorems in functions of a complex variable,

* (74).

† See (78), however.

an immediate proof could be given; and then he went on to show that by elimination of these theorems, the proof could be given a second more fundamental form and finally a third form due to Gauss and involving only distinctly elementary theorems. In a second paper* he simplified Gauss's proof very considerably by replacing Gauss's auxiliary function xf'/f by $1/f$. Here $f = 0$ is the given equation.

Here and elsewhere he succeeded in simplifying an apparently definitive proof. This kind of work was congenial to Bôcher, who believed that mathematics was capable of almost indefinite simplification, and that such simplification was of the highest consequence.

In the paper with the title "A problem in statics and its relation to certain algebraic invariants" (1904) he employed a dynamical method similar to his extension of the method of Stieltjes in order to develop an interpretation of the roots of covariants as the positions of equilibrium of particles in the complex plane. Thus if f_1, f_2 are polynomials of the same degree in the homogeneous variables x_1, x_2 , the vanishing of their Jacobian determines the points of equilibrium in the field of force under the inverse first power law due to particles of "mass" 1 at the roots of f_1 and of "mass" -1 at the roots of f_2 in the x_1/x_2 plane.

We shall not refer to his geometrical papers† save to mention the one entitled "Einige Sätze über projective Spiegelung" (1893) in which he proves that conics in different planes may be projectively reflected into each other through a pair of lines in four ways, and also that the general collineation of space may be represented as the product of a rigid motion and a projective reflection through a pair of lines.

Besides this original research he undertook various more or less didactic articles with characteristic unselfishness.‡ However, just as in the article on Fourier's series, matter of an original cast is nearly always present.

The same may be said of his books,§ even of the most elementary. We have already considered his book on the series of potential theory. Of the others, the most significant are his Algebra, where a satisfactory exposition of the elementary

* (18).

† (6), (8), (12), (13), (53).

‡ (14), (20), (24), (39), (66), (67), (70), (73), (83), (92).

§ (15), (71), (77), (94), (95), (99).

divisor theory is given, his Cambridge tract on integral equations,* and his Paris 1913-14 lectures "Leçons sur les Méthodes de Sturm." In the last is given the first complete discussion of the convergence of the series used in the method of successive approximations. This furnishes another good instance of Bôcher's power to seize on important theorems which have been missed although near at hand. In concluding this brief survey it is worth while noting that a few of his papers are fairly popular in character.†

In a recent one of these, "Mathématiques et mathématiciens Français" (1914), while speaking of the characteristics of American creative work in all fields (page 9), Bôcher says "Ce qu'il y a de plus caractéristique dans la meilleure production intellectuelle américaine, c'est la finesse et le contrôle voulu des moyens et des effets. La faute la plus commune dans ce que nous avons fait de mieux, ce n'est pas l'excès de force, mais plutôt son défaut" and later (page 10) "Ce que je viens de dire se rapporte aussi bien aux mathématiques qu'à toute autre branche de la production intellectuelle en Amérique." There can be no doubt that this characterization is applicable to his own mathematical production. His papers excel in simplicity and elegance, and nearly all of them treat subjects of great importance to marked advantage. The *usefulness* of his papers is exceptional,‡

In amount and quality his production exceeds that of any American mathematician of earlier date in the field of pure mathematics.

Because of this fact and the weight he has added to our mathematical traditions in other ways, Maxime Bôcher will ever remain a memorable personality in American mathematics.

LIST OF BÔCHER'S WRITINGS.||

1888.

- (1) The meteorological labors of Dove, Redfield and Espy. *American Meteorological Journal*, vol. 5, No. 1, pp. 1-13, May.

* In connection with this, attention should be called to a short note on integral equations listed as (84) below.

† (1), (9), (11), (82), (90), (91). His first paper "On the meteorological labors of Dove, Redfield and Espy" was a youthful essay written about the time of his graduation from Harvard University.

‡ This is brought out clearly in Professor Osgood's *Lehrbuch der Funktionentheorie*, vol. 1.

|| Substantially as compiled by him.

1891.

- (2) Über die Reihenentwickelungen der Potentialtheorie. Gekrönte Preisschrift und Dissertation. Göttingen, Kästner. 4 + 66 pp.

1892.

- (3) On Bessel's functions of the second kind. *Annals of Mathematics*, vol. 6, No. 4, pp. 85–90, Jan.
(4) Pockels on the differential equation $\Delta u + k^2 u = 0$ [Review]. *Annals of Mathematics*, vol. 6, No. 4, pp. 90–92, Jan.
(5) Geometry not mathematics [Letter to editor]. *Nation*, vol. 54, No. 1390, p. 131, Feb.
(6) On a nine-point conic. *Annals of Mathematics*, vol. 6, No. 5, p. 132, March.
(7) On some applications of Bessel's functions with pure imaginary index. *Annals of Mathematics*, vol. 6, No. 6, pp. 137–160, May.
(8) Note on the nine-point conic. *Annals of Mathematics*, vol. 6, No. 7, p. 178, June.
(9) Collineation as a mode of motion. *Bulletin of the New York Mathematical Society*, vol. 1, No. 10, pp. 225–231, July.

1893.

- (10) On the differential equation $\Delta u + k^2 u = 0$. *American Journal of Mathematics*, vol. 15, No. 1, pp. 78–83, Jan.
(11) A bit of mathematical history. *Bulletin of the New York Mathematical Society*, vol. 2, No. 5, pp. 107–109, Feb.
(12) Some propositions concerning the geometric representation of imaginaries. *Annals of Mathematics*, vol. 7, No. 3, pp. 70–72, March.
(13) Einige Sätze über projective Spiegelung. *Mathematische Annalen*, vol. 43, No. 4, pp. 598–600.
(14) Chapter IX, Historical Summary, pp. 267–275. *An Elementary Treatise on Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics*. By W. E. Byerly. Boston, Ginn.

1894.

- (15) Über die Reihenentwickelungen der Potentialtheorie. Mit einem Vorwort von Felix Klein. Leipzig, Teubner, 8 + 258 pp.

1895.

- (16) Hayward's Vector Algebra [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, No. 5, pp. 111–115, Feb.
(17) Gauss's third proof of the fundamental theorem of algebra. *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, No. 8, pp. 205–209, May.
(18) Simplification of Gauss's third proof that every algebraic equation has a root. *American Journal of Mathematics*, vol. 17, No. 3, pp. 266–268, July.
(19) General equation of the second degree [Set of formulas on a card]. Harvard University Press.

1896.

- (20) On Cauchy's theorem concerning complex integrals. *Bulletin of the American Mathematical Society*, ser. 2, vol. 2, No. 5, pp. 146–149, Feb.
(21) Bessel's functions [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 2, No. 8, pp. 255–265, May.

- (22) Linear differential equations and their applications. [Report by T. S. Fiske of a lecture at the Buffalo Colloquium]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 2, pp. 52-55, Nov.
- (23) Heffter's Linear Differential Equations [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 2, pp. 86-92, Nov.
- (24) Regular points of linear differential equations of the second order. Cambridge, Harvard University Press, 23 pp.

1897.

- (25) Schlesinger's Linear Differential Equations [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 4, pp. 146-153, Jan.
- (26) On certain methods of Sturm and their application to the roots of Bessel's functions. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 6, pp. 205-213, March.
- (27) Review of Bailey and Woods: Plane and Solid Analytic Geometry. *Bulletin of the American Mathematical Society*, ser. 2, vol. 3, No. 9, pp. 351-352, June.

1898.

- (28) Examples of the construction of Riemann's surfaces for the inverse of rational functions by the method of conformal representation. By C. L. Bouton with an introduction by Maxime Bôcher. *Annals of Mathematics*, vol. 12, No. 1, pp. 1-26, Feb.
- (29) The roots of polynomials which satisfy certain linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 6, pp. 256-258, March.
- (30) The theorems of oscillation of Sturm and Klein (first paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 7, pp. 295-313, April.
- (31) The theorems of oscillation of Sturm and Klein (second paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 8, pp. 365-376, May.
- (32) Note on some points in the theory of linear differential equations. *Annals of Mathematics*, vol. 12, No. 2, pp. 45-53, May.
- (33) Note on Poisson's integral, *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 9, pp. 424-426, June.
- (34) Niewengłowski's Geometry [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 4, No. 9, pp. 448-452, June.
- (35) The theorems of oscillation of Sturm and Klein (third paper). *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 1, pp. 22-43, Oct.

1899.

- (36) Burkhardt's Theory of Functions [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 4, pp. 181-185, Jan.
- (37) On singular points of linear differential equations with real coefficients. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 6, pp. 275-281, March.
- (38) An elementary proof that Bessel's functions of the zeroth order have an infinite number of real roots. *Bulletin of the American Mathematical Society*, ser. 2, vol. 5, No. 8, pp. 385-388, May.
- (39) Examples in the theory of functions. *Annals of Mathematics*, ser. 2, vol. 1, No. 1, pp. 37-40, Oct.

1900.

- (40) On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 40-52, Jan.; also No. 4, p. 507, Oct.

- (41) Some theorems concerning linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 6, No. 7, pp. 279–280, April.
- (42) Application of a method of d'Alembert to the proof of Sturm's theorems of comparison. *Transactions of the American Mathematical Society*, vol. 1, No. 4, pp. 414–420, Oct.
- (43) On linear dependence of functions of one variable. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 3, pp. 120–121, Dec.
- (44) Randwertaufgaben bei gewöhnlichen Differentialgleichungen. *Encyklopädie der mathematischen Wissenschaften*, II A 7a, pp. 437–463, Leipzig, Teubner.

1901.

- (45) The theory of linear dependence. *Annals of Mathematics*, ser. 2, vol. 2, No. 2, pp. 81–96, Jan.
- (46) Green's functions in space of one dimension. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 7, pp. 297–299, April.
- (47) Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence. *Transactions of the American Mathematical Society*, vol. 2, No. 2, pp. 139–149, April.
- (48) An elementary proof of a theorem of Sturm. *Transactions of the American Mathematical Society*, vol. 2, No. 2, pp. 150–151, April.
- (49) Non-oscillatory linear differential equations of the second order. *Bulletin of the American Mathematical Society*, ser. 2, vol. 7, No. 8, pp. 333–340, May.
- (50) On certain pairs of transcendental functions whose roots separate each other. *Transactions of the American Mathematical Society*, vol. 2, No. 4, pp. 428–436, Oct.
- (51) On Wronskians of functions of a real variable. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 2, pp. 53–63, Nov.
- (52) Picard's *Traité d'Analyse* [Review]. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 3, pp. 124–128, Dec.

1902.

- (53) Some applications of the method of abridged notation. *Annals of Mathematics*, ser. 2, vol. 3, No. 2, pp. 45–54, Jan.
- (54) Review of Schlesinger: *Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen*. *Bulletin of the American Mathematical Society*, ser. 2, vol. 8, No. 4, pp. 168–169, Jan.
- (55) On the real solutions of two homogeneous linear differential equations of the first order. *Transactions of the American Mathematical Society*, vol. 3, No. 2, pp. 196–215, April.
- (56) On systems of linear differential equations of the first order, *American Journal of Mathematics*, vol. 24, No. 4, pp. 311–318, Oct.
- (57) Review of Gauss' *Wissenschaftliches Tagebuch*. *Bulletin of the American Mathematical Society*, ser. 2, vol. 9, No. 2, pp. 125–126, Nov.

1903.

- (58) *The Elements of Plane Analytic Geometry*. By George R. Briggs. Revised and enlarged by Maxime Bôcher. New York, Wiley, 4 + 191 p.
- (59) Singular points of functions which satisfy partial differential equations of the elliptic type. *Bulletin of the American Mathematical Society*, ser. 2, vol. 9, No. 9, pp. 455–465, June.

- (60) On the uniformity of the convergence of certain absolutely convergent series. *Annals of Mathematics*, ser. 2, vol. 4, No. 4, pp. 159–160, July.

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- (61) Contribution to Sprechsaal für die Encyklopädie der Mathematischen Wissenschaften. *Archiv der Mathematik und Physik*, vol. 7, No. 3, p. 181, Feb.
- (62) The fundamental conceptions and methods of mathematics. Address delivered before the Department of Mathematics of the International Congress of Arts and Science, St. Louis, Sept. 20, 1904. *Bulletin of the American Mathematical Society*, ser. 2, vol. 11, No. 3, pp. 115–135, Dec. Also in Congress of Arts and Science Universal Exposition, St. Louis, 1904, vol. 1. Boston, Houghton and Mifflin, 1905, pp. 456–473.
- (63) A problem in statics and its relation to certain algebraic invariants. *Proceedings of the American Academy of Arts and Sciences*, vol. 40, No. 11, pp. 469–484, Dec.

1905.

- (64) Linear differential equations with discontinuous coefficients. *Annals of Mathematics*, ser. 2, vol. 6, No. 3, pp. 97–111 (49–63), April.
- (65) Sur les équations différentielles linéaires du second ordre à solution périodique. *Comptes Rendus de l'Académie des Sciences*, vol. 140, No. 14, pp. 928–931, April.
- (66) A problem in analytic geometry with a moral. *Annals of Mathematics*, ser. 2, vol. 7, No. 1, pp. 44–48, Oct.

1906.

- (67) Introduction to the theory of Fourier's series. *Annals of Mathematics*, vol. 7, No. 2, and No. 3, pp. 81–152, Jan. and April.
- (68) On harmonic functions in two dimensions. *Proceedings of the American Academy of Arts and Sciences*, vol. 41, No. 26, pp. 577–583, March.
- (69) Review of Picard: Sur le Développement de l'Analyse, etc. *Science*, n. s., vol. 23, No. 598, p. 912, June.
- (70) Another proof of the theorem concerning artificial singularities. *Annals of Mathematics*, ser. 2, vol. 7, No. 4, pp. 163–164, July.

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- (71) Introduction to Higher Algebra. By Maxime Bôcher. Prepared for publication with the cooperation of E. P. R. Duval. New York, Macmillan, 11 + 321 pp.*

1908.

- (72) Review of Bromwich: Quadratic Forms and their Classification by Means of Invariant Factors. *Bulletin of the American Mathematical Society*, ser. 2, vol. 14, No. 4, pp. 194–195, Jan.
- (73) On the small forced vibrations of systems with one degree of freedom. *Annals of Mathematics*, ser. 2, vol. 10, No. 1, pp. 1–8, Oct.

1909.

- (74) On the regions of convergence of power-series which represent two-dimensional harmonic functions. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 271–278, April.

* A German translation appeared in 1909: Einführung in die höhere Algebra. Deutsch von Hans Beck. Mit einem Geleitwort von Eduard Study. Leipzig, Teubner, 12 + 348 pp.

- (75) Review of Runge: *Analytische Geometrie der Ebene*. *Bulletin of the American Mathematical Society*, ser. 2, vol. 16, No. 1, pp. 30–33, Oct.
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ON A THEOREM OF OSCILLATION.

BY PROFESSOR WILLIAM F. OSGOOD.

IN his book on series which represent the potential function Bôcher makes use of the first part of the following theorem.* All the variables are real.

THEOREM—*Let $\varphi(t)$ be continuous and monotonic in the interval $T \leq t < \infty$, and let $\varphi(t)$ be always greater numerically than a certain positive constant γ :*

$$|\varphi(t)| > \gamma.$$

If $\varphi(t) < 0$, an arbitrary solution of the differential equation

$$\frac{d^2y}{dt^2} = \varphi(t) y,$$

oscillates an infinite number of times in any interval $T \leq T' \leq t < \infty$, in which it is considered, the amplitudes of the oscillations remaining finite.

If, furthermore, $\varphi(t)$ remains finite, the amplitudes of the oscillations do not become less than a certain positive constant. Moreover, the amplitudes vary monotonically, increasing when $\varphi(t)$ increases algebraically, and decreasing in the opposite case.

Bôcher states the theorem without the restriction that $\varphi(t)$ be monotonic, and outlines a suggestive dynamical proof. Professor Birkhoff's comment in the foregoing article led me to examine critically both theorem and proof. It is easy to see† that the theorem is not true under Bôcher's hypotheses. He needed the theorem, however, only in the restricted form above given. Moreover, he does not state the last paragraph of the theorem, this extension not being requisite for his purposes.

As regards this extension, it is not difficult to show that, if $\varphi(t)$ is not required to remain finite, the amplitudes of the oscillations may become infinitesimal, when t becomes infinite.

* Ueber die Reihenentwickelungen der Potentialtheorie; Teubner, 1894, p. 178.

† Cf. infra.

Proof. Let

$$p = p(t) = -\varphi(t), \quad p_n = p(t_n).$$

To integrate the differential equation

$$(1) \quad \frac{d^2 y}{dt^2} = -py,$$

proceed in the usual manner:

$$2 \frac{dy}{dt} \frac{d^2 y}{dt^2} = -2py \frac{dy}{dt},$$

$$\left(\frac{dy}{dt} \right)^2 = -2 \int py dy + C.$$

We will now introduce with Bôcher the dynamical interpretation of the equation (1) as representing Newton's second law of motion for a particle of unit mass acted on by a central attractive force whose intensity, in absolute units, is py , and moving in a right line through the center of force O . Here, t represents the time, and y the distance, measured algebraically, of the particle from O .

We will determine the constant of integration for two particular cases.

Case A. The particle is projected from O in the sense of the increasing y with an initial velocity $v = c$. Then

$$\left(\frac{dy}{dt} \right)^2 = c^2 - 2 \int_0^y py dy.$$

The amplitude h of this half-oscillation is given [by putting $dy/dt = 0$, $y = h$. Hence

$$(2) \quad \int_0^h py dy = \frac{c^2}{2}.$$

Case B. The particle is released from rest at the point $y = h > 0$. Then

$$\left(\frac{dy}{dt} \right)^2 = 2 \int_y^h py dy.$$

The velocity c with which it reaches the center of force is given by the equation

$$c^2 = 2 \int_0^h py dy,$$

which is of the same form as (2).

Thus both cases lead to the same formal relation (2) between c and h .

Consider, now, an arbitrary solution y of (1) in the interval $T \leq t' \leq t < \infty$, and suppose

$$y|_{t=t'} > 0.$$

Since the force is attractive and its intensity is at least γy , the particle will be pulled in to the center of force, and the given solution will vanish at a definite instant* $t = t_0 > t'$.

It is sufficient, then, to consider a half-oscillation, i. e., an excursion from the center of force back to that point. Let the particle be projected at the instant $t = t_0$ from O with velocity v_0 in the direction of the positive y 's.

I. Let the monotonic function $p(t)$ decrease. Then

$$p(t) > \gamma, \quad p(\infty) \geq \gamma.$$

Corresponding to Case A, we have for the motion away from O to rest

$$(3) \quad \frac{p_1 h_1^2}{2} \leq \uparrow \int_0^{h_1} py dy = \frac{v_0^2}{2}.$$

Hence

* The argument can be equally well carried through analytically. The curve corresponding to the solution in question, $y = f(t)$, is concave downward, since $d^2y/dt^2 < 0$. If $f'(t') > 0$, the curve must have a maximum and return to the initial level, $y = f(t') = b$. For, while it is above that level,

$$d^2y/dt^2 = -py < -\gamma b = -k.$$

Consider the parabola for which $d^2y/dt^2 = -k$, and which is tangent initially to the above curve:

$$(i) \quad Y = b + f'(t')(t - t') - \frac{1}{2}k(t - t')^2.$$

Then the given curve will lie below the parabola so long as it lies above the line $y = b$. For, by Taylor's theorem with the remainder,

$$(ii) \quad y = b + f'(t')(t - t') + \frac{1}{2}f''(\tau)(t - t')^2, \quad t' < \tau < t.$$

Hence

$$Y - y = \frac{1}{2}[py - k]_{t=\tau} (t - t')^2,$$

and the bracket is positive so long as $y \geq b$.

Thus the curve $y = f(t)$ cuts the line $y = b$ a second time, and at this point the slope is negative. From now on the curve lies below its tangent at this point, till it cuts the axis of t .

$$h_1 \leq \frac{v_0}{\sqrt{p_1}} \leq \frac{v_0}{\sqrt{\gamma}}.$$

Corresponding to Case B, we have for the motion back* to O

$$(4) \quad \frac{p_1 h_1^2}{2} \geq \downarrow \int_0^{h_1} p y dy = \frac{v_2^2}{2}.$$

From (3) and (4) we infer that

$$v_2 \leq v_0.$$

It thus appears that, as the motion continues, the successive values $v_0, v_2, v_4, \dots$ never increase, and hence h_n remains finite, since

$$h_{n+1} \leq \frac{v_{2n}}{\sqrt{\gamma}} \leq \frac{v_0}{\sqrt{\gamma}}.$$

To show that h_n remains greater than a certain positive constant, we replace (3) and (4) by the new relations

$$(5) \quad \frac{p_0 h_1^2}{2} \geq \uparrow \int_0^{h_1} p y dy = \frac{v_2^2}{2};$$

$$(6) \quad \frac{p_2 h_1^2}{2} \leq \downarrow \int_0^{h_1} p y dy = \frac{v_2^2}{2}.$$

Hence,

$$h_1 \geq \frac{v_0}{\sqrt{p_0}}, \quad \frac{v_2}{\sqrt{p_2}} \geq \frac{v_0}{\sqrt{p_0}}.$$

On writing down the corresponding relations for the further half-oscillations and taking v_{2m} always to mean the numerical value of dy/dt at O ,

$$\frac{v_4}{\sqrt{p_4}} \geq \frac{v_2}{\sqrt{p_2}},$$

.....

$$\frac{v_{2n}}{\sqrt{p_{2n}}} \geq \frac{v_{2n-2}}{\sqrt{p_{2n-2}}},$$

we obtain

$$h_{n+1} \geq \frac{v_{2n}}{\sqrt{p_{2n}}} \geq \frac{v_0}{\sqrt{p_0}},$$

* p , considered as a function of y , is multiple-valued; but in each of the cases A and B we are dealing only with a single-valued branch of the function.

and thus the theorem is proven for the case of a decreasing function $p(t)$, the corresponding function $\varphi(t)$ increasing.

Finally, h_n increases monotonically,

$$h_{n+1} \geq h_n,$$

since from (5) and (6)

$$\frac{v_0}{\sqrt{p_0}} \leq h_1 \leq \frac{v_2}{\sqrt{p_2}},$$

etc.

II. When $p(t)$ is an *increasing* function, the proof is given in a similar manner. Instead of (3) and (4) we now have

$$(7) \quad \frac{p_0 h_1^2}{2} \leq \int_0^{h_1} p y dy = \frac{v_0^2}{2};$$

$$(8) \quad \frac{p_2 h_1^2}{2} \geq \int_0^{h_1} p y dy = \frac{v_2^2}{2}.$$

Here, we have

$$h_1 \leq \frac{v_0}{\sqrt{p_0}}, \quad \frac{v_2}{\sqrt{p_2}} \leq \frac{v_0}{\sqrt{p_0}}.$$

Hence

$$h_{n+1} \leq \frac{v_{2n}}{\sqrt{p_{2n}}} \leq \frac{v_0}{\sqrt{p_0}},$$

and h_n remains finite.

To show that h_n does not fall below a positive constant, write

$$(9) \quad \frac{p_1 h_1^2}{2} \geq \int_0^{h_1} p y dy = \frac{v_0^2}{2};$$

$$(10) \quad \frac{p_1 h_1^2}{2} \leq \int_0^{h_1} p y dy = \frac{v_2^2}{2}.$$

Hence

$$h_1 \geq \frac{v_0}{\sqrt{p_1}} \geq \frac{v_0}{\sqrt{\gamma}}, \quad v_2 \geq v_0,$$

and thus

$$h_{n+1} \geq \frac{v_0}{\sqrt{\gamma}}.$$

From (7) and (8) it follows that

$$\frac{v_2}{\sqrt{p_2}} \leq h_1 \leq \frac{v_0}{\sqrt{p_0}},$$

etc.

Hence

$$h_{n+1} \leq h_n,$$

and h_n decreases monotonically.

Physical Interpretation.—A physical interpretation of the problem just discussed is given by the small oscillations of a simple pendulum, the cord of which, in Case I, is gradually lengthened, monotonically, remaining finite; while in Case II the cord is monotonically shortened, never becoming less than a positive fixed length.

The same physical picture is useful, too, for showing that the theorem, stated without restriction on $\varphi(t)$, is false. For the way a child swings higher and higher is to lengthen the equivalent simple pendulum on the downward arc, and suddenly to shorten it as the upward arc begins.

Bôcher's theorem (b), l. c., namely that, when $\varphi(t)$ is continuous and

$$0 < g \leq \varphi(t) \leq G,$$

there is one and only one solution which remains finite, and this vanishes at infinity, is correct without any further restriction on the function φ . An elementary analytic proof can be given by the aid of the law of the mean and Taylor's theorem with the remainder, carried through the term of the second order.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS,
December 8, 1918.

PROOF OF A PROPERTY OF THE NORM OF A CYCLOTOMIC INTEGER.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society April 27, 1918.)

KUMMER in his first proof of the general law of reciprocity between two ideals in a regular cyclotomic algebraic field gave a theorem* which forms a link in his chain of reasoning. Let

* *Abhandlungen Berlin Academy*, 1859, p. 119, formula (7).

$$\omega(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{p-2}x^{p-2},$$

$$\omega = a_0 + a_1\alpha + \cdots + a_{p-2}\alpha^{p-2},$$

where $a_0, a_1, \dots, a_{p-2}$ are integers, p is a prime, $\alpha = e^{2\pi i/p}$ and ω is prime to p . Write

$$D_s(\omega) = \left[\frac{d^s \log \omega(e^v)}{dv^s} \right]_{v=0},$$

and let $N(\omega)$ be the norm of ω . Then we shall prove that

$$(1) \quad D_{p-1}(\omega) \equiv \frac{\omega(1)^{p-1} - N(\omega)}{p} \pmod{p},$$

which result was given by Kummer in the article cited. Kummer's proof depended on certain expansions in logarithmic series. In the present paper we proceed from another point of view. Put

$$N(\omega) = \omega(\alpha)\omega(\alpha^2)\cdots\omega(\alpha^{p-1}).$$

In this relation α may be replaced by an indeterminate x if we add a suitable multiple of $x^{p-1} + x^{p-2} + \cdots + 1$. Letting $x = e^v$, we may then write

$$(2) \quad \omega(e^v)\omega(e^{2v})\cdots\omega(e^{(p-1)v}) = N(\omega) + \frac{e^{vp} - 1}{e^v - 1} X,$$

where X is a rational integral function of e^v . Put $v = 0$ in this relation; we obtain

$$(3) \quad pX(1) + N(\omega) = (\omega(1))^{p-1}.$$

Taking logarithms in (2), differentiating $(p-1)$ times with respect to v , and setting $v = 0$ we have, if $V = (e^{vp} - 1)/(e^v - 1)$,

$$(4) \quad \begin{aligned} & (1 + 2^{p-1} + 3^{p-1} + \cdots + (p-1)^{p-1})D_{p-1}(\omega) \\ & \equiv \left[\frac{d^{p-2}}{dv^{p-2}} \left(\frac{1}{N + VX} \frac{d(VX)}{dv} \right) \right]_{v=0} \\ & \equiv \left[\frac{1}{N + VX} \frac{d^{p-1}(VX)}{dv^{p-1}} \right]_{v=0} \pmod{p}, \end{aligned}$$

since $D_s(V) \equiv 0 \pmod{p}$ for $S < p-1$ and

$$[N + VX]_{v=0} \not\equiv 0 \pmod{p},$$

ω and therefore N being prime to p . We have further

$$(5) \quad \left[\frac{d^{p-1}(VX)}{dv} \right]_{v=0} \equiv X(1) \left[\frac{d^{p-1}V}{dv} \right]_{v=0} \equiv -X(1) \pmod{p}.$$

Comparison of (5), (4) and (3) gives the theorem (1).

PHILADELPHIA,
April, 1918.

TRAJECTORIES AND FLAT POINTS ON RULED SURFACES.

BY MR. J. K. WHITEMORE.

(Read before the American Mathematical Society April 28 and October 27, 1917.)

§1. *Introduction.* In the following paper we determine the flat points* of a ruled surface with real rulings, and prove a new property of the orthogonal trajectories of the rulings. This property may be extended to any isogonal trajectory of the rulings, not itself a ruling, and may be regarded as a generalization of Bonnet's familiar theorem.†

Let S be a ruled surface with real rulings, g any such ruling, C an orthogonal trajectory of the rulings, generally not a straight line; let the coordinates of any point of C be x_0, y_0, z_0 , and consider these as functions of v , the arc of C . When C is not a straight line let the direction cosines of its tangent be α, β, γ , of its principal normal be l, m, n , and of its binormal be λ, μ, ν ; let R and T be respectively the radii of curvature and torsion of C , ψ the angle measured from the principal normal towards the binormal to the direction chosen as positive on g . We suppose C to be a rectifiable curve generally without singular points in the portion considered; the curvature $1/R$ and the torsion $1/T$ shall have finite first derivatives with respect to v , and ψ shall have a finite second derivative.

The surface S is given by

$$(1) \quad x = x_0 + uL, \quad L = l \cos \psi + \lambda \sin \psi,$$

* Flat points are defined in § 4 of this paper.

† See Eisenhart, *Differential Geometry*, p. 248.

with similar equations for y and z , where u is the length measured from C on g in the positive direction. If C is a straight line S is a right conoid. To give its equations we choose C as the X axis and let ψ be the angle made by the positive direction on g measured from the Y axis towards the Z axis. We have

$$(1') \quad x = v, \quad y = u \cos \psi, \quad z = u \sin \psi.$$

The linear element of (1) is given by

$$(2) \quad ds^2 = du^2 + [1 - 2uq + u^2(p^2 + q^2)]dv^2$$

$$p = \frac{1}{T} - \psi' \quad q = \frac{\cos \psi}{R}.$$

For (1') ds is also given by (2) making $1/R = 1/T = 0$. S is developable if p is identically zero; if q is also identically zero S is a plane or a cylinder.

It is well known* that the linear element of S may be given the form

$$(3) \quad ds^2 = du_1^2 + [(u_1 - a)^2 + b^2]dv_1^2,$$

where the curves (v_1) , that is v_1 constant, are the rulings, u_1 is the length measured along a ruling from an orthogonal trajectory of the rulings, $u_1 = 0$, a depending on v_1 alone the distance from this trajectory to the central point of the ruling so that $u_1 = a$ is the equation of s , the line of striction of S ; b also depending on v_1 alone is the parameter of distribution of S . The total curvature K of S from (2) and (3) is

$$K = \frac{-p^2}{[1 - 2uq + u^2(p^2 + q^2)]^2} = \frac{-b^2}{[(u_1 - a)^2 + b^2]^2}.$$

Comparing (2) and (3), we have

$$a = \frac{q}{p^2 + q^2} \quad b = \frac{p}{p^2 + q^2},$$

the sign of b being determined in the usual way.†

§ 2. *A property of the trajectories.* Suppose a point v of C is on s ; for this point $a = 0$, and $q = \cos \psi/R = 0$. If $1/R = 0$ for all points of C that curve is a straight line, a

* Eisenhart, p. 247.

† Eisenhart, p. 246.

geodesic, and by Bonnet's theorem is s ; if $1/R = 0$ for the point v , but not for all points of C , it follows that C is at this point tangent to the asymptotic line of S , not g , through v , unless the point v is a flat point. If $1/R \neq 0$, $\cos \psi = 0$ and the normal to S at v coincides with the principal normal of C , and the geodesic curvature of C vanishes at the point. If at a point of intersection of C and s the former is tangent to an asymptotic line its principal normal is not necessarily normal to S , as may be seen from the following construction: let C be any curve satisfying the conditions of § 1, v a point of C for which $1/R = 0$, $1/T \neq 0$, let S be a surface of normals to C , ψ having any *constant* value. The point v is on s since $q = 0$, $p = 1/T \neq 0$; S has at this point an ordinary point which is not a flat point since $K \neq 0$. The curve C is tangent to an asymptotic line at v , and the angle of the normal to S and the principal normal of C at v is the complement of ψ .

Suppose conversely that v is a point of C where the geodesic curvature vanishes; then $\cos \psi = 0$, $q = 0$, and, unless $p = 0$, $a = 0$, and v is on s . If $p = 0$, $K = 0$ for v and consequently for all points of the ruling through v . Such a ruling we shall call a *zero ruling*. In the next section we examine the behavior of orthogonal trajectories of the rulings at points of intersection with zero rulings.

The properties of orthogonal trajectories just proved may be extended without modification to all isogonal trajectories—a ruling itself not being considered, however, as a trajectory. We may state the results of this section in

Theorem 1: At a point of intersection of an isogonal trajectory of the rulings of a surface S with the line of striction the geodesic curvature of the trajectory vanishes unless the tangent to the latter is asymptotic or the point is a flat point of S ; conversely, a point of a trajectory where the geodesic curvature vanishes is either on the line of striction or on a zero ruling.

§ 3. *Trajectories on zero rulings.* If the total curvature K vanishes for v and a finite value of u , $p = 0$; either $b = 0$, or a or b or both become infinite. If $q \neq 0$, $b = 0$ and $a = 1/q$ is finite; if $q = 0$ either a or b becomes infinite. We distinguish two cases of zero rulings: (I) $p = 0$, $q \neq 0$; (II) $p = q = 0$. That this is a distinction independent of the trajectory C appears from the corresponding properties of a and b , as also from properties of the point of the spherical indicatrix of S corresponding to the ruling, which are stated below.

Let D, D', D'' be the coefficients of the second fundamental form of S ; D is identically zero, and $K = 0$ gives $D' = 0$. The last condition gives

$$|LL'\alpha| = 0 \quad L' = -\alpha q + (l \sin \psi - \lambda \cos \psi)p.$$

For (I)

$$\frac{\alpha}{L'} = \frac{\beta}{M'} = \frac{\gamma}{N'} = -\frac{1}{q}.$$

For (II)

$$L' = M' = N' = 0.$$

In (I) the point of the spherical indicatrix of S corresponding to v , that is the point of coordinates (L, M, N) , is a simple point of the indicatrix, and the tangent to the indicatrix at v is parallel to the tangent to C at v . The orthogonal trajectory intersecting a zero ruling (I) on s has at the point of intersection a singular point, as may be seen in various ways: (1) $a = 0$ gives $R = 0$; (2) the tangent to the spherical indicatrix at a point corresponding to a ruling g is parallel to the normal to S at the point of intersection of g and s ,* but we have seen that the tangent to the indicatrix is parallel to the tangent to the orthogonal trajectory, a contradiction in the case of an ordinary point of the trajectory on s . This remark proves incidentally that the tangent to any other orthogonal trajectory at a point of intersection with a zero ruling (I) is parallel to the normal to S at the intersection of the ruling with s . Since in (I) $\cos \psi \neq 0$ no orthogonal trajectory has zero geodesic curvature at a point on the ruling.

In (II) $q = 0$ gives $\cos \psi = 0$ or $1/R = 0$; if $\cos \psi = 0$ the trajectory has zero geodesic curvature; if $1/R = 0$ the point is a flat point of S , for the ruling has the only asymptotic direction at a point for which $K = 0$. The behavior of a trajectory at a flat point will be considered in the following section. To a zero ruling (II) corresponds a singular point of the spherical indicatrix since

$$L' = M' = N' = 0.$$

An example of a zero ruling (II) is furnished by the surface of binormals of a curve C at a point for which the torsion vanishes. In this example b is infinite and a may have any value.

* Eisenhart, p. 351. This does not apply to a developable surface.

§ 4. *Flat points.* A flat point of a surface is a point where the curvature of every normal section is zero; it may also be defined as an umbilic where the total curvature of the surface vanishes.* The conditions for an umbilic are, in the usual notation,

$$\frac{D}{E} = \frac{D'}{F} = \frac{D''}{G};$$

for a flat point $D = D' = D'' = 0$. For S , with the coordinates u, v used in the preceding sections, $D = 0, E = 1$, so that such a surface has no umbilics other than flat points. It is worth while at this point to emphasize the requirement that S be a surface with *real* rulings, for there exists a real ruled surface with no flat points all of whose real points are umbilics—the sphere.

To determine the flat points of a surface S we have, in addition to the identity $D = 0$ and the condition for zero total curvature $D' = 0$, from which v may be found, the equation for $u, D'' = 0$, which gives

$$(4) \quad u^2 |LM'N''| + u \{ |x_0''MN'| - |x_0'MN''| \} + |x_0''Mz_0'| = 0,$$

in which each bracket of three letters represents a determinant with principal diagonal in evidence.

For a zero ruling (I) equation (4) has two roots,

$$u = -\frac{1}{\sqrt{\Sigma L'^2}} = \frac{1}{q} \quad u = -\frac{|x_0''MN'|}{|LM'N''|} = \frac{\sin \psi}{Rt}$$

$$t = \frac{\sin \psi \cos \psi}{R^2} - p'.$$

The first value gives the central point of the ruling which is generally not a flat point but is a singular point of the coordinate curve (u), the orthogonal trajectory through the point. The second value gives the same point if $p' = 0, \sin \psi \neq 0$; if $p' = \sin \psi = 0, D'' = 0$ for all u 's, and all points of the ruling are flat points; if $p' \neq 0, t \neq 0$, the general case, there is one flat point; if $p' \neq 0, t = 0$ there is no flat point.

For a zero ruling (II) the coefficient of u^2 in (4) vanishes, and that equation gives for a flat point one value,

* Scheffers, *Theorie der Flächen*, 2d ed., pp. 126, 146.

$$u = \frac{|x_0'' M z_0'|}{|x_0' M N''|} = - \frac{\sin \psi}{R p'}.$$

The ruling has one finite flat point if $p' \neq 0$; if $p' = 0$ there is no flat point unless also $1/R = 0$; if $p' = 1/R = 0$ all points of the ruling are flat points for $D'' = 0$ for all u 's. The case of the right conoid, where C is a straight line, is included in (II), for since $1/R = 0$, $q = 0$; one flat point is on C , and all other or no other points of the ruling are flat points according as $p' = -\psi''$ does or does not vanish.

In (I) we have $a \cos \psi = R$, so that the projection of the central point of the ruling on the osculating plane of any orthogonal trajectory C , satisfying the requirements of § 1, is the center of curvature of the corresponding point of the trajectory. In the case of a single flat point in (I) suppose for the moment C is the orthogonal trajectory through that point and that C satisfies the requirements of § 1; we have

$$u = \frac{\sin \psi}{R t} = 0.$$

Since a is finite $1/R \neq 0$, and $\sin \psi = 0$, and the ruling is the principal normal of the trajectory through the flat point, and the central point of the ruling is the corresponding center of curvature of that trajectory. If in (I) all points of the ruling are flat points $\sin \psi = 0$ for any orthogonal trajectory satisfying the requirements of § 1; the ruling is the principal normal and the central point the center of curvature of all such trajectories. The latter case is illustrated by taking for S a plane regarded as the locus of tangents of one of its curves s . It may be proved if $p' = 0$, $\sin \psi \neq 0$, no flat point, that the principal normals of all orthogonal trajectories at points of the zero ruling are parallel. This case is illustrated on any ruling of a developable surface whose points are not flat points. In the case remaining in (I), $p' \neq 0$, $t = 0$, no flat point, there is no simple relation between the ruling and the trajectories.

In (II) the ruling is the binormal of every orthogonal trajectory not intersecting the ruling at a flat point. If there is a single flat point on the ruling, ψ may have any value at this point; if all points of the ruling are flat points there is no simple relation between the ruling and the trajectories.

By a discussion similar to the preceding we may prove analogous properties of isogonal—not orthogonal—trajectories of the rulings of S . We choose such a trajectory as the direc-

trix C , and suppose C cuts the rulings at the angle $A \neq 0$. No trajectory has zero geodesic curvature at a point of intersection with a zero ruling (I). Every trajectory through the central point of a ruling (I) has at this point a singular point, but if $A \neq \pi/2$ the coordinate curve (u) through the central point has not a singular point at this point, so that the point is generally not a singular point of the surface. This case is again illustrated by a plane regarded as the locus of the tangents of one of its curves. Every trajectory intersecting a zero ruling (II) at a point not a flat point has zero geodesic curvature at this point.

We consider finally the behavior of the flat points when a surface S is deformed into a surface of the same sort with correspondence of rulings.* From (2) it follows that in such deformation the values of p^2 and q are unchanged for any point of C ; therefore a zero ruling remains such and moreover remains always in the same case, (I) or (II). Since the arc v is also unchanged the same is true of the absolute value of p' , but $1/R$ and ψ are generally changed, so that the flat point moves on the ruling during deformation. In (I) deformation cannot change a ruling of one flat point into a ruling of all flat points, but can change either of these into a ruling with no flat point. In (II) a ruling of one flat point remains such during deformation, but a ruling of no flat points may be changed into a ruling of all flat points. A developable surface never has a ruling with a single flat point. The deformations for rulings of all and no flat points are illustrated for (I) by a plane regarded as the locus of tangents of one of its curves, for (II) by a cylinder. If all points of a surface are flat points the surface is a plane.† We remark that all cases considered in this section actually occur, for a surface S may be constructed to satisfy the requirements of § 1 with values arbitrarily assigned to $1/R$, ψ , p , p' at a point of C .

We state the chief results of this section in

Theorem 2: On a zero ruling of a surface S there is generally one flat point. The geodesic curvature of an isogonal trajectory of the rulings at a point of intersection with a zero ruling, not a flat point, is zero for all trajectories or for none.

YALE UNIVERSITY, 1918.

* If two ruled surfaces are applicable the rulings correspond necessarily unless both surfaces are applicable to the same quadric. See Eisenhart, p. 347.

† Scheffers, p. 146.

POINTS OF VIEW OF CAUCHY AND WEIERSTRASS
IN THE THEORY OF FUNCTIONS.

Leçons sur les Fonctions monogènes uniformes d'une Variable complexe. Par EMILE BOREL. Rédigées par GASTON JULIA. Gauthier-Villars, Paris, 1917. xii + 165 pp.

THE point of departure in this monograph is the opposition between the points of view of Cauchy and Weierstrass in the definition of functions of a complex variable called *monogenic* by Cauchy and *analytic* by Weierstrass. The conclusion which is reached—and it is supported by detailed investigation and significant results—is that the point of view of Cauchy is the more fundamental one. In fact, it is contended that the work of Weierstrass is confined to a class of functions more special than that which should be considered and that in a more general class the central theorems are maintained, so that there is no reason for confining attention entirely to the analytic functions of Weierstrass.

The researches of which we have here an exposition convenient for the reader go back as far as a quarter century. Borel entered upon some of the matters involved as early as 1892 and published the first results in 1894. But it was as late as 1912 when he made known the principal results of the theory of non-analytic monogenic functions. This was in his lecture before the Fifth International Congress of Mathematicians at Cambridge. This earlier exposition was incomplete in many respects; and it was only in the course of lectures at Paris of which this book is the reproduction that he has completely developed the theory, entering into detail in the demonstrations and leaving no delicate point without careful illumination.

It was intended that this book should appear at the end of 1914; but M. Gaston Julia, a student at the Ecole Normale Supérieure, who was charged with preparing the notes for publication, was called into the army and was wounded in January, 1915. Notwithstanding his sufferings, he continued with the labor of seeing the manuscript through the press even though he was also at the same time engaged in remarkable researches of his own suitable "for perpetuating French mathematical traditions." The difficulties under which the

proofreading was done probably account for the rather large number of minor misprints, more than a score of which the reviewer detected. None of them is such as to cause the reader serious inconvenience.

The intimate connection between the theory of functions of a complex variable and the theory of the potential in mathematical physics has been an important guide to Borel in his researches on non-analytic monogenic functions, as he has pointed out in his Cambridge lecture; he has been helped particularly by the analogies of the new theory with the theories of molecular physics so much in consideration among scientists in very recent years. The non-analytic monogenic functions correspond to the case where the singular regions are at the same time extremely small and extremely numerous.

In the monograph under consideration attention has been confined to what may be called the purely mathematical theory of monogenic functions, and none of the physical applications just indicated are developed. In his preface the author states that it had originally been his intention to publish simultaneously with this monograph another in which were to be given the applications of these theories to the problems of molecular physics. But he finds himself compelled to leave off this study "until the moment when the victory of civilization . . . over a brutal aggression shall give us the right to take up again the thoughts and labors of times of peace." Some indication of what he has in mind in this further study is to be found in the second note in the monograph under review and in the seventh note in his *Introduction géométrique à quelques Théories physiques*. Perhaps one must wait until the appearance of a completed exposition of these applications before reaching a final judgment on the matters insisted upon by Borel; but the development already before us seems to indicate that the point of view of Cauchy is indeed more fundamental than that of Weierstrass and in a way which will require our thinking of the latter simply as the most important and most elegant case of the former.

Cauchy has laid down the conditions of monogeneity which lie at the base of the theory of functions of a complex variable and in his integral has created a matchless instrument of demonstration for dealing with this theory. In the study of differential equations he employed the process of analytic continuation. The rôle of Weierstrass was to make more pre-

cise certain points of the latter theory, notably that which pertains to the form of the domains. We may speak of the domains of Weierstrass, or the domains W , to mean those domains in which an analytic function may be defined by the process of analytic continuation as employed by Weierstrass. These domains may be defined geometrically and their properties be developed without reference to functions which are or may be defined in them.

The first chapter of this monograph (pages 1–23) is devoted to certain generalities (pages 1–5), the definition of a monogenic function (pages 5–10), geometric properties of domains W (pages 10–18), and the representation of an analytic function defined in a domain W (pages 19–23), the latter being given in the form of a Cauchy integral with the consequent series of Taylor and Laurent. Thus the chapter deals throughout with classic matter; the discussion of domains W may be singled out as particularly pleasing.

Chapter II (pages 24–54) is devoted to an application of the integral of Cauchy to the development in a series of polynomials of a function defined in a domain of Weierstrass. The first fourteen pages are given to an exposition of the method of Runge and the remainder of the chapter to applications to certain particular developments.

The third chapter (pages 55–72) is devoted to certain remarkable consequences of the development in series of polynomials and to an extension of the theory of analytic continuation. Here we have some preliminary aspects of Borel's reasons for insisting upon the inadequacy of the Weierstrass theory from the point of view of the more general theory of monogenic functions, the arguments here being presented in such way as to enable one to pass to the conclusion as directly as possible from first considerations. Then, after a necessary preliminary treatment of sets of points of measure zero (in Chapter IV, pages 73–124), he introduces in Chapter V (pages 125–150) the domains C , or domains of Cauchy, and develops the first elements of the general theory of non-analytic monogenic functions defined in these domains. There is nothing to indicate that these are the most general domains in which monogenic functions may be defined; but they are more general than the domains of Weierstrass.

Poincaré, with great ingenuity, constructed analytic expressions presenting certain remarkable singularities and be-

lieved that he had sufficient grounds for concluding to the impossibility of extending the theory of analytic functions beyond the bounds fixed by Weierstrass; that is, he believed in the impossibility of a process of analytic continuation by which one could extend a function beyond the domain of Weierstrass. With this negative result Borel has been for a long time in disagreement, at first thinking that it was not established and now finally justifying his position by means of the detailed results set forth in this monograph. Particularly, he has objected to the conclusion that the domain throughout which a function could be continued analytically by means of power series necessarily constitutes its *natural* domain of existence, that the function does not exist outside of this domain, and that every attempt to pierce this Unknown is destined to failure and contradiction. In this connection it is interesting to note the following from the preface to this monograph:

“A la suite des travaux de Poincaré que j’ai rappelés il y un instant, ce point de vue paraissait universellement admis; mais tandis que Poincaré accueillait avec bienveillance le premier essai dans lequel je montrais que les limites imposées par Weierstrass n’étaient pas aussi infranchissables qu’on l’avait cru, les disciples fidèles de Weierstrass ne consentaient même pas à discuter; je me rappellerai toujours l’étonnement avec lequel je vis M. Mittag-Leffler, auquel j’avais essayé d’exposer mes projets de recherches, ne faire aucun effort pour entrer dans ma pensée et se contenter de retirer de sa malle un Mémoire de Weierstrass pour me montrer une phrase qui devait clore définitivement toute discussion: Magister dixit.”

In Chapter III we have an interesting example bearing on this matter of dispute and one which seems to settle the question definitively in favor of Borel’s contention. Let $a_1, a_2, a_3, \dots$ be the rational numbers between 0 and 1 arranged in enumerable sequence and consider the infinite series

$$(1) \quad f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - e^{2\pi i a_n}}, \quad i = \sqrt{-1},$$

where the A ’s are constants such that the series $A_1 + A_2 + \dots$ converges absolutely. It is easy to see that this series defines a function which is holomorphic at every point in the interior of the unit circle about the point zero. It may be shown that this circle is a *natural* boundary, in the sense of

Weierstrass, of the function defined by the series at an interior point of this circle. Likewise the series defines a function which is holomorphic at every point exterior to this unit circle and the circle is a natural boundary of the function thus defined. In the theory of Weierstrass there is no means of continuation by which one may establish a connection between the function defined by (1) in the interior of the unit circle and that so defined in the exterior of this circle. In fact, from the point of view of Weierstrass they are to be treated as unrelated functions.

Nevertheless Borel shows that a suitable continuation does exist for establishing the connection between these functions, this being constituted by a series of polynomials. In fact, one can form a series of polynomials

$$\sum_{p=0}^{\infty} P_p(z)$$

which converges uniformly on every finite segment of every straight line of argument $m\sqrt{2} + n$, where m and n are integers and $m \neq 0$, an infinity of which lines lie in every angle issuing from the origin; on each of these lines the series converges to the sum of the series in (1).

R. D. CARMICHAEL.

SHORTER NOTICE.

Elements of Optics for the Use of Schools and Colleges. By GEORGE W. PARKER, M. A. London, New York, and Bombay, Longmans, Green and Company, 1915. vi + 122 pp.

THE mathematical prerequisites necessary for reading this little book have been reduced to a minimum. A student whose knowledge of mathematics is limited to an acquaintance with elementary geometry, the solution of simple algebraic equations and a few fundamental propositions in trigonometry will be able to follow the treatment at all places. The knowledge of physical phenomena presupposed is also reduced to an extreme minimum. The book is therefore of a strictly elementary character. It is written in a satisfactory style and its material is arranged in interesting sequence, so that it may be

recommended to one who seeks pleasing applications of the most elementary mathematics to a chapter in scientific theory.

The chief merits of the exposition as an elementary treatment of its subject matter are intimately dependent upon the straightforward and simple manner of presentation on account of which the reader is able to follow the development with striking unity of effort and with little loss of energy consumed through divergent operations of thought. This renders the book particularly valuable for the learner who needs to concentrate attention upon the main issues in order to understand them thoroughly.

The effort to attain the advantages just mentioned has also led the author into the chief defects of his exposition. These are associated with the description of a special case as though it were the general case. Thus a lens is defined (page 56) as "a transparent body bounded by two spherical surfaces" and the student is left without any hint that lenses may also be of other forms. The most usual form of the kaleidoscope is described (page 13) as if there were no other form. A similar defect is in such a definition as that of optics (page 1) as "the science which treats of the properties of this mysterious agent" light, whereas the book itself deals with only a very narrow range of the properties of light and the student is given no hint of the more fundamental matters not treated in the book. The mathematical reader also feels a certain uneasiness in the free use of "infinity" (as on pages 84, 89, 113, and elsewhere) and in the uncritical use of processes of approximation. Nevertheless these minor defects do not obscure the real interest and value of this very elementary exposition.

R. D. CARMICHAEL.

NOTES.

THE fourth annual meeting of the Mathematical Association of America was held at the University of Chicago on Friday, December 27, 1918, in connection with the annual meeting of the American Mathematical Society. The morning programme included a conference on "Deductions from war time experiences with respect to the teaching of mathematics," a paper on "An experiment in supervised study," by D. R.

CURTISS, and the election of officers. At the afternoon joint session with the American Mathematical Society the following papers were presented: "Some mathematical features of ballistics," by A. A. BENNETT; "How the map problem was met in the war," by KURT LAVES; "Notes concerning recent books on navigation," by ALICE B. GOULD; "Statistics methods for preparation for war department service," by H. L. RIETZ; "Ordnance problems," by W. D. MACMILLAN; "Practical exterior ballistics," by P. L. ALGER; "The effect of the earth's rotation and curvature on the path of a projectile," by W. H. ROEVER; "On low velocity high angle fire," by H. F. BLICHFELDT. The evening was devoted to a joint dinner of the two organizations at the Quadrangle Club.

THE December number (volume 20, number 2) of the *Annals of Mathematics* contains the following papers: "The gamma function in the integral calculus (concluded)," by T. H. GRONWALL; "Invariants which are functions of parameters of the transformation," by O. E. GLENN; "A theorem on exhaustible sets connected with developments of positive real numbers," by HENRY BLUMBERG; "Solution of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and its application to some geometrical problems," by ALEXANDER PELL; "A general method of summation of divergent series," by L. L. SMAIL.

THE fourth volume (1918) of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers:—number 5 (May): "The structure of an electromagnetic field," by H. BATEMAN; "Invariants which are functions of parameters of the transformation," by O. E. GLENN; number 7 (July): "On the representation of a number as the sum of any number of squares, and in particular of five or seven," by G. H. HARDY; number 8 (August): "Arithmetical theory of certain Hurwitzian continued fractions," by D. N. LEHMER; "On closed curves described by a spherical pendulum," by ARNOLD EMCH; number 9 (September): "On the α -holomorphisms of a group," by G. A. MILLER; number 10 (October) "Invariants and canonical forms," by E. J. WILCZYNSKI; number 11 (November): "On certain projective generalizations of metric theorems, and the curves of Darboux and Segre," by G. M. GREEN; number 12 (December): "On Jacobi's extension of the continued fraction algorithm," by

D. N. LEHMER; "A characterization of Jordan regions by properties having no reference to their boundaries," by R. L. MOORE.

THE Council of the Mathematical Association of America has appointed as Committee on Publications Professors R. C. ARCHIBALD (editor in chief) W. A. HURWITZ and H. E. SLAUGHT. This Committee has charge of the official journal of the Association, the *American Mathematical Monthly*.

AT the annual meeting of the Paris academy of sciences, held December 2, 1918, the following prizes in pure and applied mathematics were awarded, in addition to those noted in the December number of the BULLETIN: The Grand prize, for a memoir on the theory of iteration, to G. JULIA, with honorable mention for the late S. LATTÈS, who was also awarded a prize of 2000 francs from the Gegner foundation for his work in mathematical analysis; a prize of 2000 francs from the Jérôme Ponti foundation to P. BARBARIN for his work in non-euclidean geometry; a prize of 2000 francs from the Henri Becquerel foundation to P. FATOU for his work on the theory of series and the iteration of rational functions. The Fourneyron, Damoiseau and Guzman prizes were not awarded.

All the prizes are to be awarded in 1919 and following years under the usual conditions. For the Bordin prize (3000 francs), to be awarded in 1921, the following question is proposed: "To improve the theories relating to analysis situs developed in Poincaré's celebrated memoirs. It is required to connect, at least in important special cases, questions of the geometry of situation concerning a given multiplicity with the study of suitably chosen analytic expressions."

DR. F. W. REED has been appointed instructor in mathematics at the University of Illinois.

MR. A. D. CAMPBELL and DR. C. M. SMITH have been appointed instructors in mathematics at Cornell University.

THE resignation of Lieutenant W. E. MILNE, of the Army Ordnance, has been accepted and he has been appointed assistant professor of mathematics at the University of Oregon.

MR. J. S. MIKESH has been appointed instructor in mathematics at Sheffield Scientific School, Yale University.

DR. R. JENTZCH, of the University of Berlin, fell in battle, March 21, 1918.

PROFESSOR W. B. STONE, of Rutgers College, died January 6, 1918. Professor Stone had been a member of the American Mathematical Society since 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BATEMAN (H.). Differential equations. London, Longmans, 1918. 8vo. 11+306 pp. 16s.
- BOYER (P.). La vie universitaire à Paris. Paris, Colin, 1918. Fr. 12.00
- FORSYTH (A. R.). Solutions of the examples in a treatise on differential equations. London, Macmillan, 1918. 8vo. 249 pp. 10 s.
- GALLATLY (W.). The modern geometry of the triangle. 2d edition. London, F. Hodgson, 1918. 8vo. 7+126 pp. 2s. 6d.
- HEDRICK (H. B.). Interpolation tables or multiplication tables of decimal fractions giving the products to the nearest unit of all numbers from 1 to 100 by each hundredth from 0.01 to 0.99 and of all numbers from 1 to 1000 by each thousandth from 0.001 to 0.999. Washington, Carnegie Institution, 1918. Folio. 9+139 pp. \$5.00
- LAUGEL (L.). See SCHWARZ (H. A.).
- PLANT (L. C.). See WEBBER (W. P.).
- SANSONE (G.). Le divisioni regolari dello spazio iperbolico in piramidi e doppie piramidi. Pisa, tip. succ. fratelli Nistri, 1917. 8vo. 135 pp. con quattro tavole.
- SCHWARZ (H. A.). Mélanges relatifs au domaine des surfaces minima. Traduit sur la dernière édition par L. Laugel. Pisa, E. Spoerri, 1918. 8vo. 53 pp. L. 4.00
- WEBBER (W. P.) and PLANT (L. C.). Introductory mathematical analysis. New York, Wiley, 1918. 250 pp. \$2.00

II. ELEMENTARY MATHEMATICS.

- ABBOTT (P.). Mathematical tables and formulæ. London, Longmans, 1918. 4+58 pp. 2s.
- . Numerical trigonometry. London, Longmans, 1918. 10+163+mathematical tables, 3+33 pp. 5s.
- CADORIUS (I.), FISKE (A.), HERLEV (V.) og OLSEN (L. P.). Regnebogen, Elevens Bog IV, V og VI. Lærerens Bog III, IV og V. København, Gyldendalske Boghandel, 1918.
- FERRARI (A.). Sopra un capitolo di geometria elementare. Torino, S. Lattes, 1918. 8vo. 32 pp.

FISKER (A.). See CADORIUS (I.).

FRIIS-PETERSEN (F.) og JESSEN (J. L. W.). Realklassens Regnebog. København, Gjellerup, 1918.

GLADDEN (T. L.). Plane geometry. Boston, R. G. Badger, 1918. 165 pp. \$1.00

HERLEV (V.). See CADORIUS (I.).

HOPKINS (J. W.) and UNDERWOOD (P. H.). Arithmetic. Book 1, revised. New York, Macmillan, 1918. 12mo. 11+260 pp. \$0.40

JESSEN (J. L. W.) og SMITH (O. A.). Matematik og Regning for Realklassen. København, Gjellerup, 1918.

——. See FRIIS-PETERSEN (F.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy, the Royal Military College; and papers in elementary engineering for naval cadetships. Nov., 1917 and March, 1918. With answers. London, Macmillan, 1918. 40 pp. 1s. 3d.

MINNICK (J. H.). An investigation of certain abilities fundamental to the study of geometry. (Diss., University of Pennsylvania.) Lancaster, Pa., New Era Press, 1918. Sm. 4to. 8+108 pp. \$1.25

NIELSEN (N.). Lærebog i Trigonometri. 2den gennemste Udgave. København og Kristiania, Gyldendalske Boghandel, 1917.

OLSEN (L. P.). See CADORIUS (I.).

SMITH (O. A.). See JESSEN (J. L. W.).

UNDERWOOD (P. H.). See HOPKINS (J. W.).

II. APPLIED MATHEMATICS.

ANDOYER (H.). Formules et tables nouvelles relatives à l'étude du mouvement des comètes et à différents problèmes de la théorie des orbites. Paris, Gauthier-Villars, 1918. 8vo. 52 pp. Fr. 3.00

BONASSE (H.). Astronomie théorique et pratique. (Bibliothèque scientifique de l'ingénieur et du physicien.) Paris, Delagrave, 1918. 8vo. 30+630 pp. Relié. Fr. 35.00

BRITAIN (W. M.). Bibliography of maritime literature; published by the American steamship association for distribution among the officers and crews of steamers operated by its members. New York, 1918. 8vo. 14 pp.

KEMPNER (A. J.). Simple applications of trigonometry to artillery. Urbana, Ill., University of Illinois, 1918. 8vo. 30 pp. \$0.25

LÉVY (M.). La statique graphique et ses applications aux constructions. 3e partie: Arcs métalliques. Ponts suspendus rigides. Conpoles et corps de révolution. 3e édition. Paris, Gauthier-Villars, 1918. 8vo. 10+418 pp. Fr. 24.00

LIPKA (J.). Graphical and mechanical computation. New York, Wiley, 1918. 9+264 pp. \$4.00

MAILLOT (L.). Le manuel du mécanicien. Métaux usuels, trempe, recuit, ajustage, tournage, filetage, fraisage, meulage, outils de mesure et de vérification. 3e édition. Paris, Dunod et Pinat, 1918. 8vo. Fr. 3.50

- MANN (C. R.). A study of engineering education prepared for the joint committee on engineering education of the national engineering societies. (Bulletin No. 11.) New York, Carnegie Foundation for the Advancement of Teaching, 1918. 4to. 11+139 pp.
- MARGUET (F.). Cours d'astronomie de l'Ecole Navale. Paris, Challamel, 1918. 8vo. Fr. 10.00
- O'DONNELL (E. E.). Merchant marine manual. Boston, Yachtman's Guide, 1918. 254 pp. \$1.00
- PALMER (C. I.). Practical mathematics. 2d edition. New York, McGraw-Hill, 1918. 4 volumes. 152+172+188+150 pp. \$0.75 each
- POOR (C. L.). Simplified navigation for ships and aircraft. A text-book based upon the Saint Hilaire method. New York, Century, 1918. 18+126 pp.
- RUST (A.). Practical tables for navigators and aviators, containing new and rapid methods for finding the longitude, azimuth and latitude and for great circle sailing, the identification of stars and for plotting line of position by the Sumner and Marcq Saint Hilaire methods. Philadelphia, J. E. Hand and Sons Company, 1918. 116 pp. \$3.50
- SALMON (V. G.). Practical surveying and field work. London, C. Griffin and Company, 1918. 7s. 6d.
- SAMPSON (R. A.). Studies in clocks and time-keeping. No. 2: Tables of the circular equation. Edinburgh, Grant, 1918. 2s. 10d.
- WARNE (E. J.). Elementary course in cartography: twenty lessons, seventy charts. Washington, F. J. Warne, 1917. Paper. \$50.00

Publications of the American Mathematical Society

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Mathematical Papers of the Chicago Congress, 1893.

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and EDWARD KASNER. Price, \$1.50; to members, \$1.00.

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MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

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THE TWENTY-FIFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth annual meeting of the Society was held in Chicago on Friday and Saturday, December 27-28, 1918. The attendance included the following sixty members:

Professor R. C. Archibald, Professor C. S. Atchison, Professor R. P. Baker, Captain A. A. Bennett, Professor H. F. Blichfeldt, Dr. H. Blumberg, Professor J. W. Bradshaw, Professor H. T. Burgess, Professor W. D. Cairns, Professor J. A. Caparo, Professor R. D. Carmichael, Professor H. E. Cobb, Professor A. B. Coble, Professor C. E. Comstock, Mr. H. W. Curjel, Professor D. R. Curtiss, Professor L. E. Dickson, Professor E. L. Dodd, Mr. E. B. Escott, Professor W. B. Ford, Professor Tomlinson Fort, Professor O. E. Glenn, Miss A. B. Gould, Dr. J. O. Hassler, Professor Olive C. Hazlett, Professor T. H. Hildebrandt, Professor L. C. Karpinski, Professor A. M. Kenyon, Professor K. Laves, Professor S. Lefschetz, Professor A. C. Lunn, Professor Gertrude I. McCain, Professor M. McNeill, Professor W. D. MacMillan, Professor Bessie I. Miller, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Mr. E. E. Moots, Professor E. J. Moulton, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaughter, Professor G. W. Smith, Professor E. B. Stouffer, Professor E. J. Townsend, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor F. B. Wiley, Professor R. E. Wilson, Professor C. H. Yeaton.

At the first session, held Friday afternoon, with Professor Curtiss in the chair, President L. E. Dickson presented his retiring address on "Mathematics in War Perspective." Following this address there was a joint session of the Society with the Mathematical Association of America at which President Dickson presided. The programme at this session was devoted to mathematical problems in connection with the war. The following papers were presented:

(1) Captain A. A. BENNETT: "Some mathematical features of ballistics."

(2) Professor KURT LAVES: "How the map problem was met in the war."

(3) Miss ALICE B. GOULD: "Notes concerning recent books on navigation."

(4) Professor H. L. RIETZ: "Statistics methods for preparation for war department service."

(5) Major W. D. MACMILLAN: "Ordnance problems."

(6) Lieutenant P. L. ALGER: "Practical exterior ballistics."

(7) Professor W. H. ROEVER: "The effect of the earth's rotation and curvature on the path of a projectile."

(8) Professor H. F. BLICHFELDT: "On low velocity high angle fire."

Brief accounts of these papers will be published in the report of the meeting by the Secretary of the Association in the *American Mathematical Monthly*.

On Friday evening the two organizations held a joint dinner at the Quadrangle Club, attended by sixty-eight members and friends.

At the session of the Society on Saturday, President Dickson presided in the morning and Professor Curtiss in the afternoon. The Council, which met Saturday morning, made the following announcements: There was one election to membership in the Society, Mr. V. S. Mallory, graduate student at Columbia University; and eight new applications for membership were received. The Treasurer's report was accepted, having been examined by the auditing committee. It shows a balance of \$9,965.28, including the life membership fund of \$6,843.46. On recommendation of the committee appointed to arrange for a summer meeting and colloquium at Chicago in 1919, it was voted to postpone that meeting for one year. The usual summer meeting, without colloquium, will be held in 1919 at a place to be determined.

At the annual election which closed Saturday noon, there were one hundred and fifty-nine votes cast. The following officers and other members of the Council were chosen:

President, Professor FRANK MORLEY.

Vice-Presidents, Professor G. D. BIRKHOFF,
Professor FLORIAN CAJORI.

Secretary, Professor F. N. COLE.

Treasurer, Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1921,

Professor H. E. HAWKES, Professor A. C. LUNN,
Professor W. A. HURWITZ, Professor C. N. MOORE.

The following papers were presented at the Saturday sessions:

(1) Professor W. B. FORD, "The sum of any series expressed as a definite integral with application to analytic continuation."

(2) Professor HARRIS HANCOCK: "On the foundations of the elliptic functions."

(3) Professor E. L. DODD: "A comparison of the median and the arithmetic mean of measurements for various laws of error."

(4) Professor DANIEL BUCHANAN: "Asymptotic orbits near the straight line equilibrium points in the problem of three finite bodies."

(5) Professor G. A. MILLER: "Groups containing a small number of sets of conjugate operators."

(6) Professor OLIVE C. HAZLETT: "A theorem on modular covariants."

(7) Mr. WAYNE SENSENIG: "Concerning the covariant theory of involutions of conics."

(8) Professor C. H. SISAM: "On surfaces containing two pencils of cubic curves."

(9) Professor E. B. STOUFFER: "On singular ruled surfaces in S_5 ."

(10) Dr. HENRY BLUMBERG: "A property of linear point sets."

(11) Professor E. J. WILCZYNSKI: "An application of line geometry to the theory of functions" (preliminary communication).

(12) Professor C. N. MOORE: "On the integro-derivative and some of its applications."

(13) Dr. I. A. BARNETT: "Differential equations with a continuous infinitude of variables."

(14) Dr. BARNETT: "Linear integro-differential equations with constant kernels."

(15) Dr. A. R. SCHWEITZER: "On iterative functional equations of the distributive type."

(16) Professor R. P. BAKER: "Valuation of railroad ties."

(17) Professor BAKER: "Asymptotic forms in probability."

(18) Professor HARRIS HANCOCK: "Rational and integral expressions for the roots of the biquadratic."

(19) Professor ARNOLD EMCH: "On a certain class of rational ruled surfaces."

(20) Dr. W. G. SIMON: "Two formulas of interpolation in two variables."

(21) Professor J. B. SHAW: "On triply orthogonal congruences."

(22) Professor S. LEFSCHETZ: "Real folds of abelian varieties."

(23) Dr. T. H. GRONWALL: "On surfaces of constant curvature with an equation of the form $u(x) + v(y) + w(z) = 0$."

(24) Dr. GRONWALL: "A theorem on power series with application to conformal mapping."

(25) Dr. GRONWALL: "Equipotential minimal surfaces."

(26) Dr. GRONWALL: "On Kummer's series."

(27) Professor P. J. DANIELL: "Integrals in an infinite number of dimensions."

(28) Professor C. N. MOORE: "Generalized limits in general analysis."

(29) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (fourth paper)."

Mr. Sensenig was introduced by Professor Glenn, and Dr. Barnett by Professor Wilczynski. Professor Hancock's second paper was read by Professor C. N. Moore. In the absence of the authors, the papers by Professor Buchanan, Mr. Sensenig, Professor Sisam, Professor Emch, Dr. Simon, Dr. Gronwall, Professor Daniell, and the first paper by Professor Hancock were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By a method based on the calculus of residues, Professor Ford obtains an expression, in the form of a double improper integral, for the sum of any (convergent) series. When applied in particular to a power series, this expression preserves

a meaning (in general) for values of the variable lying outside the circle of convergence and thus furnishes the analytic continuation of the function defined by the series. The paper is related to a former paper of the author published in *Liouville's Journal* in 1903.

2. The differential equations through which $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ or $\wp u$, $\sqrt{\wp u - e_1}$, $\sqrt{\wp u - e_2}$, $\sqrt{\wp u - e_3}$ are defined may be put in the form of the differences of two logarithmic expressions. When integrated it may be shown that each of these expressions is a uniformly convergent power series. Thus in a direct and simple manner, without the introduction of auxiliary functions such as the *Al*-functions introduced in this connection by Weierstrass, Professor Hancock shows that the above functions become quotients of power series which are in fact theta functions.

3. Professor Dodd finds that with any symmetric law of error, the reliability of the median of measurements increases approximately as the square root of the number of measurements, whereas the reliability of the arithmetic mean may remain constant, as in the case of the law

$$\frac{c}{\pi} \frac{dx}{c^2 + x^2},$$

or even decrease. With certain non-Gaussian distributions, the median has a better chance than the arithmetic mean of lying close to the true value.

The so-called probability curve is at best only a good first approximation,—Karl Pearson's curves often give a much better fit. Instead of using a Chauvenet criterion to throw away measurements not conforming to this first approximation, it would appear better to ascertain more closely the law of error for a given distribution—if the number of measurements is large enough—and to base upon this a choice of arithmetic mean, median, or other function of the measurements.

4. It was shown by Lagrange that it is possible to start three finite bodies in such a way that they will describe similar ellipses. There are two configurations of equilibrium which the three bodies will maintain, viz., they are either col-

linear or lie at the vertices of an equilateral triangle. Professor Buchanan determines orbits which are asymptotic, in the sense of Poincaré, to the particular Lagrangian orbits which are described when the three bodies are collinear and move in coplanar circles.

5. The object of Professor Miller's paper is to establish a few general theorems relating to the possible number of sets of conjugate operators in a group of a given order and to determine all the possible groups having a small number of complete sets of conjugate operators. When all the operators of a finite group G appear in k sets of conjugates the order of G cannot exceed the k th term diminished by unity in the series

$$2, 3, 7, 43, 1807, \dots$$

where each term after the first is obtained by multiplying together all the preceding terms and increasing by unity the product thus obtained. In particular, a necessary and sufficient condition that a discontinuous group is of finite order is that its operators appear in a finite number of sets of conjugates under the group.

There is one and only one simple group of composite order which has the property that all of its operators appear in exactly five complete sets of conjugates, viz., the icosohedral group. The octic group, the quaternion group and the metacyclic group of order 20 are the only non-abelian groups whose operators appear in five complete sets of conjugates and whose commutator quotient group is of order 4. If all the operators besides the identity of a non-abelian group belong to two sets of conjugates under its group of isomorphisms, the order of this non-abelian group is either of the form p^a or of the form $p^a - q$, p and q being prime numbers. If a non-abelian group of order p^a has the property that all of its operators besides the identity appear in three sets of conjugates under its group of isomorphisms, then its commutator subgroup is of order $p^{a/2}$.

6. Professor Hazlett's paper answers a question which Dr. Sanderson raised in her thesis,* but did not answer. In her paper the fundamental theorem on the relation between formal and modular invariants for the $GF[p^n]$ enables us to construct modular covariants of a system S of binary forms in

* *Transactions Amer. Math. Society*, vol. 14 (1913), p. 490.

x and y from modular invariants of the same system S in the variables ξ and η and an additional linear form in ξ and η with coefficients y and $-x$. This is closely analogous to the situation in the algebraic theory of invariants. In the latter theory the converse is known to be true—that is, we get *all* covariants in this manner. In the case of modular invariants, however, we do not obtain all covariants, for the universal covariant $L = xy^{p^n} - yx^{p^n}$ can not be obtained as an invariant of a linear form as it vanishes whenever x and y are in the $GF[p^n]$, as we suppose the coefficients of our forms to be. In the paper referred to above, the question was raised as to whether all the covariants of a system S can be expressed as functions of L and the modular invariants of S and an additional linear form. The present paper proves that such is the case, and then, by induction, extends this theorem to a system of forms in n sets of binary variables which are transformed cogrediently.

7. Mr. Sensenig's paper contains the development and tabulation of the simultaneous concomitant set of the pencil of conics $a_x^2 + kb_x^2$, taken with the harmonic conic $(\alpha\beta x)^2$, the twenty forms being expressed as polynomials in k with coefficients which are rational expressions in the twenty forms in the known system of a_x^2 with b_x^2 . The set is given in general and also in normal form. The methods are those of generalized ternary transvection, and reduction by symbolical identities. There is derived in the paper, also, the complete system of $a_x^2 + kb_x^2$ taken with a third general conic c_x^2 . These forms are reduced in terms of sixty-one concomitants of a_x^2, b_x^2, c_x^2 from the system of three conics first derived by H. F. Baker.

The paper is to appear in the *American Journal of Mathematics*.

8. In a previous paper, Professor Sisam has determined the types of surfaces generated by an algebraic system of cubic curves which do not constitute a pencil. In the present paper he classifies and discusses the surfaces which contain two pencils of cubics, so that every type of algebraic surface with two or more cubic curves through each point is now determined. Each surface is represented birationally on as simple a surface as possible, the complete linear system to which the curves

corresponding to the plane or hyperplane sections belong is determined and a number of characteristic properties of the various types of surfaces are pointed out.

9. By means of a system of three ordinary linear homogeneous differential equations of the second order, Professor Stouffer studies the properties of singular ruled surfaces in space of five dimensions, including ruled surfaces in space of four dimensions. Only surfaces developable in the ordinary sense are excluded from the theory. The geometrical properties of various curves on the surface are shown, the transversal surface is obtained, and some of the relations between a surface and its transversal surface are studied.

10. Let S be any linear point set whatsoever. Let P be a point of S , i an interval enclosing P , l_i the length of i , and $m_e(i, S)$ the exterior measure (Lebesgue) of the subset of S in i . Defining the "exterior épaisseur" of S at P as the limit (if it exists) of $m_e(i, S)/l_i$ as l_i approaches 0, Dr. Blumberg proves that, except for a set of measure 0, the exterior épaisseur of S exists and is equal to 1 at every point P of S . As a consequence, every set may be represented as the sum of two sets, the first being of measure 0 and the second of exterior épaisseur 1 at every one of its points.

11. Let $w = u + iv = f(x + iy) = f(z)$ be a function of the complex variable z . Place the w plane in a position parallel to the z plane with the u and v axes parallel to the x and y axes respectively. The lines which join a point (x, y) of the z plane to the corresponding point (u, v) of the w plane form a congruence. This geometric image of a function $w = f(z)$ has been proposed by several authors, but so far no general results seem to have been obtained by following out this point of view. Professor Wilczynski has found that the congruence obtained in this way has very remarkable properties. Its focal surface is composed of two imaginary cones whose vertices are the circular points of the z plane, and whose generators correspond to the null lines of the z plane. The harmonic conjugate of the z plane with respect to the two focal cones is a real surface with real asymptotic curves whose properties have not yet been completely determined.

12. In this paper Professor C. N. Moore introduces a new generalized limit to replace the derivative of a function at points where the derivative in the ordinary sense and the so-called generalized derivative fail to exist. This limit is designated as the first integro-derivative of the r th order at the point in question and is given by the formula

$$\lim_{u \rightarrow 0} \frac{1}{u} \int_0^u dt_r \left(\frac{1}{t_r} \int_0^{t_r} dt_{r-1} \right. \\ \left. \times \left(\cdots \left(\frac{1}{t_2} \int_0^{t_2} \frac{f(x+t_1) - f(x-t_1)}{2t_1} dt_1 \right) \cdots \right) \right).$$

The utility of this conception is illustrated by means of a theorem which establishes the summability $(C, k > r + 1)$ of the derived Fourier's series to the value of the first integro-derivative of the r th order at points where this limit exists.

13. Dr. Barnett discusses the equation

$$\frac{\partial u(\xi, z)}{\partial z} = f(\xi, z, u(\xi')),$$

where ξ, ξ' are real variables in the range $(0, 1)$, z is a real variable on $|z - z_0| \leq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max |u - u_0| \leq \beta$, and $f(\xi, z, u)$ is a functional yielding for each (ξ, z, u) of the above ranges a real number. In the first part of the paper hypotheses are stated which are sufficient to insure the existence of a unique solution reducing for $z = z_0$ to an arbitrary continuous function $u_0(\xi)$ and having certain desirable continuity properties.

The solution is then considered as also depending on the initial conditions and theorems concerning the continuity and differentiability are proved. It is finally shown that there exists a solution to the equation

$$\frac{\partial g(z, u)}{\partial z} + \int_0^1 f(\xi, z, u) d_\xi g'(\xi, z, u) = 0,$$

where $g(z, u)$ is the functional to be determined, $f(\xi, z, u)$ is a known functional and $g'(\xi, z, u)$ is a functional related to the differential of $g(z, u)$ defined in the paper. The integration is taken in the sense of Stieltjes.

14. The most general solution of the integro-differential equation

$$(1) \quad \frac{\partial u(\xi, z)}{\partial z} = \int_0^1 k(\xi, \eta) u(\eta, z) dz$$

has already been given by Volterra in the form

$$u(\xi, z) = w(\xi) + \int_0^1 \sum_{\rho=0}^{\infty} \frac{z^\rho}{\rho!} k_\rho(\xi, \eta) w(\eta) d\eta,$$

where the $w(\xi)$ is an arbitrary continuous function on $(0, 1)$ and the functions k_ρ are the iterated kernels of k . Dr. Barnett shows that when the kernel k is symmetric or skew-symmetric, the solution takes on a simple form, analogous to the exponential form of the solution of a finite system of linear differential equations with constant coefficients.

The theorem for symmetric kernels is as follows:

Through the element (z_0, u_0) where z_0 is a real number and u_0 a real continuous function of ξ on $(0, 1)$ there exists one and but one solution of equation (1), viz.,

$$u(\xi, z) = u_0(\xi) + \sum_{\rho=1}^{\infty} c_\rho \varphi_\rho(\xi) (e^{(z-z_0)/\lambda_\rho} - 1)$$

where the φ_ρ are a complete normal orthogonal set of characteristic functions of k , λ_ρ the corresponding characteristic numbers and the c_ρ the Fourier coefficients of $u_0(\xi)$ with respect to the φ_ρ , i. e., $c_\rho = \int_0^1 \varphi_\rho(\eta) u_0(\eta) d\eta$.

A corresponding theorem for a special type of non-homogeneous equation is also discussed.

15. Dr. Schweitzer defines the "quasi-distributive" equations of order $k \geq 1$; an important instance is

$$(1) \quad \begin{aligned} &\phi[\phi(x_1, \dots, x_{n+1}), t_1, \dots, t_n] \\ &= \phi[x_1, \phi(x_2, t_1, \dots, t_n), \dots, \phi(x_{n+1}, t_1, \dots, t_n)] \end{aligned}$$

where $n = 1, 2, 3$, etc. For $n = 1$ this relation is the associative relation. In the domain of finite or infinite abstract groups* relation (1) is always solvable. Analytically, the

* Cf. abstract, BULLETIN, Nov., 1918, pp. 58, 59. In this abstract for $k = 3$ read $k = 2$; for $k = 2$ read $k = 1$; for interpolation read interpretation.

equation (1) has the solution

$$\phi(x_1, x_2, \dots, x_{n+1}) = \psi^{-1}[\psi(x_1) + \psi(x_2) \\ - \Theta\{\psi(x_2) - \psi(x_3), \dots, \psi(x_2) - \psi(x_{n+1})\}]$$

where Θ is an "arbitrary" function of $n - 1$ variables and ψ is an "arbitrary" function with an inverse. For $x_1 = \text{constant}$ in (1) this solution suggests immediately a particular solution of a certain properly distributive equation on two functions.

The equation (1) in connection with the equations

$$(2) \quad \begin{aligned} f[\phi(x_1, t_1, t_2, \dots, t_n), t_1, \dots, t_n] &= x_1, \\ \phi[f(x_1, t_1, t_2, \dots, t_n), t_1, \dots, t_n] &= x_1, \end{aligned}$$

yields, as an eliminant in the function f , an "identity" quasi-transitive functional equation (and conversely). This result is generalized for a finite group of quasi-transitive functional equations and thus leads to the concept of a finite group of quasi-distributive functional equations and, subsequently, to a finite group of properly distributive equations on two functions.

16. Professor Baker gives the formula

$$\sum_s [\{\sum_v \cdot \text{Coeff. } x^s m(\sum_r p_r x^{l_r})^v\} \cdot \sum_r p_r (l_r - X + s)],$$

divided by a denominator obtained from this by replacing the last $\sum_r$ by 1, for the most probable unexpired life of a railroad tie. Here p_r is the probability of a tie having a life l_r ; $\sum_r$ runs over all possible lives, $\sum_v$ from $v = [X/M]$ to $v = [X/m]$ where X is the age of the road and M, m are the maximum and minimum life of a tie. $\sum_s$ runs from $s = X - M$ to X including the first term if the result is desired immediately before the annual relaying and the last term if the result refers to a time just after this. The asymptotic form is obtained by omitting the $\sum_v$ and replacing it by 1. This holds for all positive distributions (p_r). A large enough X to use this form has not been reached; in actual cases the result oscillates about the asymptotic amount, which is greater than the text rule of "half the average life" for a uniform distribution and less than this for a skew binomial.

Some results are as follows: taking the mean average of the

two Σ 's and $m = 5$, $M = 15$, $X = 54$ to 60 inclusive, and for the two cases of uniform distribution and skew binomial $(2 + x)^{10}$, we have .

	M. A.	Asym.	Text	Oscillation
U. D.	5.266	5.25	5	1.8%
S. B.	4.037	4.032	4.167	2.2%

An oscillation of 1% is approximately \$10,000,000 for the roads of the United States.

17. Professor Baker considers the integral equations

$$(1) \quad kf(bx) = \int_{-\infty}^{\infty} f(u)f(x-u)du$$

and .

$$(2) \quad kf(bx) = \int_0^x f(u)f(x-u)du.$$

In volume 76 of the *Mathematische Annalen* Runge and Pólya have discussed the solution of

$$f(x) = \int_{-\infty}^{\infty} \phi(u)\phi(x-u)du.$$

Runge's method adds nothing to the discussion, but Pólya's makes the general solution depend on the functional equation

$$[\phi(s)]^2 = c\phi(as).$$

The resulting asymptotic forms include all of De Forest's based on polynomials, and also others, the simplest being $1/(1+x^2)$. This has an asymptotic neighborhood among continuous functions, and among double ended series, but not among polynomials. The equation (2) has no proper solution.

18. By making use of the associated realms of rationality, Professor Hancock is able to show that any root of a biquadratic may be expressed rationally and also integrally through any other root with coefficients that are rational and integral functions of the roots of the reducing cubic.

For example if $\sigma = \sqrt{p} + \sqrt{q} + \sqrt{r}$ and $\tau = \sqrt{p} - \sqrt{q} - \sqrt{r}$

the rather remarkable relation is

$$\tau = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3,$$

where the A 's are integral functions of the second degree in p .

19. It is well known that algebraic ruled surfaces may be generated by the lines joining corresponding points of an (α, β) -correspondence between the points of two algebraic curves in space. Professor Emch considers a large class of ruled surfaces defined by certain cinematic properties, which are equivalent to such a correspondence between the points of a straight line and a circle.

The parametric equations of these surfaces in cartesian coordinates are

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = (\rho - a) \cot p\theta/q,$$

where ρ and θ are the parameters, and p and q two integers which are relatively prime. It is shown that these surfaces are rational so that every plane section is a rational curve. They may be divided into two sub-classes of bifacial and unifacial surfaces according as q is even or odd.

When q is odd, the order of the surface is $2(p + q)$. In this case the directrix circle and the directrix line are respectively q -fold and $2p$ -fold curves of the surface. Besides these the surface has $\frac{1}{2}(q + 1)$ real double curves of order $4p$ or $2q$, according as $2p \geq q$, and p real double lines.

When $q = 2s$ is even, then the order of the surface is $p + q$. The directrix circle and line are s -fold and p -fold respectively. There are moreover $\frac{1}{2}(s + 1)$ or $\frac{1}{2}s$ real double curves, according as s is odd or even; they are of order $2p$ or q , according as $2p \geq q$, except when s is odd; then there are $\frac{1}{2}(s - 1)$ curves of order $2p$ and one curve of order p .

The system of points of intersection of the double curves with a plane through the directrix line may be arranged according to certain cyclic groups of various orders.

These surfaces have the remarkable property that, in certain sets, they are applicable upon each other, and their intersections with a torus lead to all so-called cyclo-harmonic curves.

It is also shown how Moebius' unifacial and bifacial bands of all orders may be cut out from these surfaces.

20. In a recent paper (*Annals of Mathematics*, volume 19,

number 4) Dr. Simon gave a formula of polynomial interpolation for a continuous function of a single real variable. The formula

$$\sigma_n(x, y) = \frac{\sum_{i=0}^n \sum_{j=0}^n f(x_i, y_j) [1 - (x_i - x)^2 - (y_j - y)^2]^n}{2 \sum_{i=0}^n \sum_{j=0}^n [1 - x_i^2 - y_j^2]^n},$$

where

$$x_i = \frac{i}{\sqrt{2n}}, \quad y_j = \frac{j}{\sqrt{2n}},$$

and the function $f(x, y)$ is defined and continuous when

$$0 < a \leq x \leq b \leq \frac{1}{\sqrt{2}}, \quad 0 < c \leq y \leq d \leq \frac{1}{\sqrt{2}},$$

is proposed in the present paper, and is the immediate extension of the earlier one to a continuous function of two real variables. Furthermore, it is shown that when the function which is being approximated satisfies a Lipschitz condition, the order of approximation is $1/\sqrt{n}$.

A trigonometric formula of approximation is also proposed for a continuous function of two real variables having the period 2π . This formula

$$\Sigma_n(x, y) = \frac{\sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1} f(x_i, y_j) \left[\cos \frac{x_i - x}{2} \cos \frac{y_j - y}{2} \right]^{2n}}{2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\cos \frac{x_i}{2} \cos \frac{y_j}{2} \right]^{2n}},$$

where the x_i and y_j are two sets of values such that $x_{i+1} - x_i = \pi/n$, $y_{j+1} - y_j = \pi/n$, is the extension of one given by Kryloff (*Bulletin des Sciences Mathématiques*, volume 41, October, 1917). Here too the order of approximation is $1/\sqrt{n}$ when the function satisfies a Lipschitz condition.

21. A congruence of curves is exemplified in the lines of a field of force, or velocity. In many questions it becomes necessary to study the rate of variation around a point of the unit vector which defines the congruence. Also in relation to the unit vector α which is tangent to the vector line (line of the congruence) at the point are other perpendicular unit vec-

tors, such as the principal normal and the binormal and others. Professor Shaw's paper is a study of the properties of such triply orthogonal sets of unit vectors. A linear vector function θ is defined which gives, for a differential displacement of the point ηds , the instantaneous axis and amount of rotation of the triply orthogonal system α, β, γ , that is $\theta(\eta) \cdot ds$. It turns out that θ is symmetric in α, β, γ and every theorem is thus true at once for each unit vector or its proper congruence. Also the curl and the divergence of each can be stated in terms of θ . The lines of equi-directed unit-tangents of the congruence are found in terms of θ . The invariant axes of θ define sets of straight lines along which the axis of rotation of the trihedral is in the direction of the displacement. Other properties of curvatures and torsions are discussed. (Cf. Rogers, *Proceedings of the Royal Irish Academy*, volume 29, section A, No. 6 (1912), pages 92-117.) Quaternion methods are used.

22. The object of Professor Lefschetz's note is to show that an abelian variety V_p of genus p and rank one can have any number of real folds represented by $2^s, s \geq p$. These folds form equivalent p dimensional cycles of V_p and their indices of connection are easily determined. There are Jacobi varieties of each of the p types obtained and their consideration yields a very simple proof of the theorem due to Harnack, according to which a curve of genus p can have at most $p + 1$ real branches. As a particular case of the above results, there are hyperelliptic surfaces of arbitrary divisor and rank one, with one, two or four real folds forming as many equivalent two dimensional cycles and each reducible by deformation to a ring.

23. In this paper, Dr. Gronwall shows that the determination of the surfaces indicated in the title reduces to the solution of the equation

$$\begin{aligned} f'(u)g'(v)h(w) + g'(v)h'(w)f(u) + h'(w)f'(u)g(v) \\ = [f(u) + g(v) + h(w)]^2 \end{aligned}$$

under the condition $u + v + w = 0$. By function theoretic means, involving an extensive use of Hadamard's theory of entire functions of finite genus, it is shown that this functional equation has only one solution, which corresponds to the surfaces of revolution of constant curvature.

24. Dr. Gronwall proves the following theorem: When the power series $w(z) = \sum_0^{\infty} a_n z^n$ converges for $|z| < 1$, and $\sum_0^{\infty} n |a_n|^2$ converges, and we write $z_1 = 1 - \rho e^{\theta i}$, $z_2 = 1 - \rho e^{\theta' i}$, where $\rho > 0$ and θ and θ' vary with ρ subject only to the conditions

$$-\frac{\pi}{2} + \epsilon \leq \theta \leq \frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{2} + \epsilon \leq \theta' \leq \frac{\pi}{2} - \epsilon,$$

then

$$w(z_1) - w(z_2) \rightarrow 0 \text{ as } \rho \rightarrow 0$$

uniformly in respect to θ and θ' . This theorem is useful in the study of the behavior of a conformal mapping function on the boundary of the mapped region.

25. The determination of all harmonic functions $V(x, y, z)$ such that $V(x, y, z) = \text{const.}$ defines a family of minimal surfaces is of a certain importance in hydrodynamics. In the present paper, Dr. Gronwall shows that beyond the known case $V = z - a \arctan y/x$ (and those derived from it by a change of coordinate axes), no such functions exist.

26. In this paper, Dr. Gronwall calls attention to the fact that many of the published proofs of the expansion

$$\begin{aligned} \log \Gamma(s) + \frac{1}{2} \log \frac{\sin \pi s}{\pi} + (s - \frac{1}{2})(\log 2\pi - c) \\ = \sum_1^{\infty} \frac{\log n}{n\pi} \sin 2n\pi s \end{aligned}$$

are deficient, and gives two new and elementary proofs.

27. In the *Annals of Mathematics*, June, 1918, Professor Daniell published a paper on "A general form of integral." By using the methods of that paper, he is now able to define integrals in an infinite number of dimensions, that is, not merely line integrals in an infinitely dimensional space. In a second part, the author defines a function which is of limited variation in an infinite number of dimensions and thereby is able to define a general Stieltjes integral.

28. Methods analogous to the methods for summing divergent series and integrals may be employed for defining derivatives of functions at points where the ordinary derivative fails to exist. This suggests building up a general theory which might be termed the theory of generalized limits in general analysis. In Professor C. N. Moore's paper a general theorem in this theory is established which includes as special cases the Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder methods for summing divergent series, the corresponding theorem for integrals due to Landau, and a third theorem with regard to the equivalence of the integro-derivatives of the Cesàro and Hölder type.

29. In a former paper Dr. Schweitzer has shown that a certain "quasi-distributive" property is satisfied by the elements of an abstract group, finite or infinite. This suggests that properly distributive functional equations have solutions in the domain of abstract groups. In verification, Dr. Schweitzer shows that the relation

$$\begin{aligned} \zeta[\tau(x_1, x_2, \dots, x_{n+1})t_1, t_2, \dots, t_m] \\ = \tau[\zeta(x_1, t_1, \dots, t_m), \dots, \zeta(x_{n+1}, t_1, \dots, t_m)] \quad m \geq 1, n \geq 1 \end{aligned}$$

has the solution (among others)

$$\begin{aligned} \tau(x_1, x_2, \dots, x_{n+1}) &= x_1 x_2^{-1} x_1 \dots x_{n+1}^{-1} x_1, \\ \zeta(x_1, t_1, \dots, t_m) &= x_1 t_1^{p_{11}} t_2^{p_{12}} t_1^{p_{13}} \dots t_m^{p_{1m}} t_1^{p_{1m}}, \end{aligned}$$

where the p 's are arbitrary integers not zero. Further, the equation on one function,

$$\tau\{x, \tau(y, z)\} = \tau\{\tau(x, y), \tau(x, z)\},$$

is satisfied by $\tau(x, y) = xy^{-1}x$. Finally, the author gives the following set of postulates (apart from closure) for an abstract group based on distributiveness:

1. $\zeta[\tau\{x, \zeta(x, y)\}, \zeta(y, z)] = \zeta(x, z)$.
2. $\zeta[\tau(x, y), z] = \tau[\zeta(x, z), \zeta(y, z)]$.
3. $\tau[x, \zeta(x, x)] = \tau[y, \zeta(y, y)]$.
4. $\tau[\zeta(x, y), x] = \zeta[\zeta(x, y), y]$.
5. $\tau[x, \tau(x, y)] = y$.

E. J. MOULTON,
Acting Secretary.

APPLICATIONS OF THE THEORY OF SUMMABILITY TO DEVELOPMENTS IN ORTHOGONAL FUNCTIONS.

BY PROFESSOR CHARLES N. MOORE.

(Read at the Chicago Symposium of the American Mathematical Society
April 12, 1918.)

THE very considerable body of literature which may be described by the above title belongs almost entirely to the present century. Its extent is only roughly indicated by the bibliography at the end of the paper, which makes no pretensions to being complete. The type of series considered here constitutes one of the three most important classes of series to which the theory of summability has been applied, the other two being power series and Dirichlet's series. A noteworthy feature of the applications with which we shall be concerned is found in their usefulness in an important branch of applied mathematics, namely the theory of the flow of heat and electricity.

§1. *The Summability of Fourier's Series.*

The first writer to deal with the topic of this section was Fejér. In his fundamental paper of 1903 [5]* he established among other results the summability (C1) of the Fourier development of an arbitrary function satisfying very wide conditions, at all points where the function is continuous or has a finite jump. We shall give a proof of this theorem, under somewhat modified conditions, which is substantially the same as Fejér's proof.

The Fourier development of $f(x)$ may be written in the form

$$(1) \quad \frac{1}{2\pi} \int_a^{2\pi+a} f(\theta) d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_a^{2\pi+a} f(\theta) \cos n(\theta - x) d\theta,$$

it being assumed that $f(x)$ is periodic of period 2π . We have for the sum of the first n terms of (1)

$$(2) \quad s_n(x) = \frac{1}{\pi} \int_a^{2\pi+a} f(\theta) \frac{\cos (n-1)(\theta - x) - \cos n(\theta - x)}{2(1 - \cos (\theta - x))} d\theta.$$

* The numbers in brackets refer to the bibliographical list at the end of the paper.

Hence for the arithmetic mean of the first n sums, we have

$$(3) \quad \frac{S_n(x)}{n} = \frac{1}{2n\pi} \int_a^{2\pi+a} f(\theta) \frac{\sin^2 \frac{n(\theta-x)}{2}}{\sin^2 \frac{\theta-x}{2}} d\theta.$$

If now we set $t = \frac{1}{2}(\theta - x)$ and then take $\alpha = x - \pi$, we obtain

$$(4) \quad \begin{aligned} \frac{S_n(x)}{n} &= \frac{1}{n\pi} \int_{-(\pi/2)}^{\pi/2} f(x+2t) \frac{\sin^2 nt}{\sin^2 t} dt \\ &= \frac{1}{n\pi} \int_0^{\pi/2} \{f(x+2t) + f(x-2t)\} \frac{\sin^2 nt}{\sin^2 t} dt. \end{aligned}$$

We are now ready for the proof of Fejér's theorem. We begin by establishing two lemmas.

LEMMA 1. *For any positive integer n , we have*

$$(5) \quad \frac{1}{n\pi} \int_0^{\pi/2} \frac{\sin^2 nt}{\sin^2 t} dt = \frac{1}{2}.$$

If we set

$$(6) \quad \begin{aligned} \sigma_n(2t) &= \frac{1}{2} + \cos 2t + \cos 4t + \cdots + \cos 2(n-1)t \\ &= \frac{\cos 2(n-1)t - \cos 2nt}{2(1 - \cos 2t)}, \end{aligned}$$

we have

$$(7) \quad \frac{\Sigma_n(2t)}{n} = \frac{\sigma_1(2t) + \sigma_2(2t) + \cdots + \sigma_n(2t)}{n} = \frac{1}{2n} \cdot \frac{\sin^2 nt}{\sin^2 t}.$$

But from (6)

$$\frac{2}{\pi} \int_0^{\pi/2} \sigma_n(2t) dt = \frac{1}{2} \quad (n = 1, 2, 3, \dots).$$

From this relationship and (7) the identity (5) readily follows.

LEMMA 2. *If $\varphi(t)$ is integrable (Lebesgue) in the interval $(0 \leq x \leq c \leq \frac{1}{2}\pi)$ and furthermore $\lim_{t \rightarrow 0} \varphi(t) = 0$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n\pi} \int_0^c \varphi(t) \frac{\sin^2 nt}{\sin^2 t} dt = 0.$$

Given an arbitrary positive ϵ , we chose $\delta < \epsilon$ and such that

$$(8) \quad |\varphi(t)| < \epsilon \quad (0 \leq t \leq \delta).$$

Then, δ being fixed, we choose m so large that

$$(9) \quad \frac{1}{n\pi \sin^2 \delta} \int_{\delta}^c |\varphi(t)| dt < \frac{\epsilon}{2} \quad (n \geq m).$$

From (8), (5), and (9) we obtain

$$\begin{aligned} \left| \frac{1}{n\pi} \int_0^c \varphi(t) \frac{\sin^2 nt}{\sin^2 t} dt \right| &\leq \frac{1}{n\pi} \int_0^{\delta} |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t} dt \\ &\quad + \frac{1}{n\pi} \int_{\delta}^c |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t} dt < \frac{\epsilon}{n\pi} \int_0^{\pi/2} \frac{\sin^2 nt}{\sin^2 t} dt \\ &\quad + \frac{1}{n\pi \sin^2 \delta} \int_{\delta}^c |\varphi(t)| dt < \epsilon \quad (n \geq m), \end{aligned}$$

which proves our lemma.

FEJÉR'S THEOREM.* *If the function $f(x)$ is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π , the series (1) will be summable (C1) to the value $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at every point for which this limit exists.*

It is obvious that this theorem can be proved by showing that

$$\lim_{n \rightarrow \infty} \left[\frac{S_n(x)}{n} - \frac{1}{2}\{f(x+0) + f(x-0)\} \right] = 0,$$

where $S_n(x)$ is defined by (3). In view of (4) and Lemma 1, the expression in brackets may be written in the form

$$\frac{1}{n\pi} \int_0^{\pi/2} \{f(x+2t) + f(x-2t) - f(x+0) - f(x-0)\} \frac{\sin^2 nt}{\sin^2 t} dt.$$

It follows at once from Lemma 2 that this last expression approaches zero as n becomes infinite. The theorem is therefore proved.†

* Fejér's original conditions are somewhat different from ours, he having required that $f(x)$ be integrable (Riemann) and become infinite at only a finite number of points. Thus his result and our result overlap. It is easy to modify our argument so as to include both results.

† With slight changes the above argument may be used to establish uniform summability throughout any interval included in an interval of continuity of $f(x)$.

Since the points of discontinuity of a function having a Riemann integral form a set of measure zero, Fejér's theorem for such functions may be stated in the form: *The Fourier development of any function having a Riemann integral is summable (C1) to the value of the function at all points except for a set of measure zero.* But Fejér's theorem does not enable us to assert anything of that sort about functions having a Lebesgue integral, since for such functions there may be no points where the limit $\frac{1}{2}[f(x+0) + f(x-0)]$ exists. However, Lebesgue has extended the second form of Fejér's theorem to functions integrable according to his definition [14], in a theorem which we shall now proceed to prove. We begin by establishing two lemmas.

LEMMA 3. *If $g(t)$ is positive or zero in the interval $(0 \leq t \leq \frac{1}{2}\pi)$, has a Lebesgue integral there, and is such that*

$$(10) \quad \lim_{t \rightarrow 0} \frac{G(t)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t g(t) dt = 0,$$

then we shall have

$$(11) \quad \lim_{k \rightarrow \infty} \int_0^{\pi/2} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} dt = 0.$$

Given an arbitrary positive ϵ , we choose a δ such that

$$(12) \quad 0 \leq \frac{G(t)}{t} < \frac{\epsilon}{4\pi} \quad (0 < t \leq \delta).$$

Then, δ being fixed, we choose m_1 , such that

$$(13) \quad \left| \int_\delta^{\pi/2} \frac{G(t)}{t^2} \cdot \sin kt dt \right| < \frac{\epsilon}{2} \quad (k \geq m_1),$$

which we may do in view of the Riemann-Lebesgue theorem with regard to the limiting values of the Fourier's constants of an integrable function.*

For values of $k > \pi/\delta$ we have, in view of (12),

$$(14) \quad \left| \int_0^{\pi/k} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} dt \right| < \frac{\epsilon}{4\pi} \int_0^{\pi/k} \frac{\sin kt}{t} dt \\ = \frac{\epsilon}{4\pi} \int_0^\pi \frac{\sin u}{u} du < \frac{\epsilon}{4}.$$

* Cf. Lebesgue, *Leçons sur les Séries trigonométriques*, p. 61.

Furthermore, on integrating by parts,

$$(15) \quad \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt \, dt = \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} \\ + \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt \, dt - \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt \, dt.$$

But

$$(16) \quad \left| \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} \right| \leq \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \cdot \frac{G(\pi/k)}{\pi/k},$$

$$(17) \quad \left| \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt \, dt \right| < \frac{\epsilon}{2\pi k} \int_{\pi/k}^{\delta} \frac{dt}{t^2} < \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2},$$

$$(18) \quad \left| \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt \, dt \right| < \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \, dt = \left[\frac{1}{k} \cdot \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} \\ + \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \, dt < \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \frac{G(\pi/k)}{\pi/k} + \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2}.$$

Combining (15), (16), (17) and (18), and taking into account (12), it is readily seen that we may choose m_2 such that

$$(19) \quad \left| \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt \, dt \right| < \frac{\epsilon}{4} \quad (k \geq m_2).$$

If we designate by m the greatest of the three quantities m_1 , π/δ and m_2 , we have from (13), (14) and (19)

$$\left| \int_0^{\pi/2} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} \, dt \right| < \epsilon \quad (k \geq m).$$

Our lemma is therefore proved.

LEMMA 4. *If $f(x)$ is integrable (Lebesgue) in the interval $(a \leq x \leq b)$, $|f(x) - f(x_0)|$, where $(a \leq x_0 \leq b)$, is for $x = x_0$ the derivative of its indefinite integral, except perhaps at a set of points of measure zero.*

We know from a well known theorem in the theory of Lebesgue integrals that $f(x) - \alpha$, where α is any constant, will, under the conditions of our lemma, be the derivative of its indefinite integral for all points of $(a \leq x \leq b)$ except perhaps a set of measure zero. Let this set be represented by $E(\alpha)$, and let E_1 represent the set which is the sum of all the $E(\alpha)$ for all rational values of α . Then E_1 will also be a set of measure zero.

Let x_0 be a value of x in $a \leq x \leq b$ which does not belong to E_1 , and let β be any irrational number. Given an arbitrary positive ϵ , we can find a rational number α so near to β that

$$(20) \quad \left| |f(x) - \beta| - |f(x) - \alpha| \right| \leq |\alpha - \beta| < \frac{\epsilon}{3},$$

whence

$$(21) \quad \left| \frac{1}{|x - x_0|} \left| \int_{x_0}^x |f(x) - \beta| dx \right| - \frac{1}{|x - x_0|} \left| \int_{x_0}^x |f(x) - \alpha| dx \right| \right| < \frac{\epsilon}{3}.$$

Since, moreover, the second term of the left hand member of (21) approaches $|f(x) - \alpha|$ as x approaches x_0 , we may choose a δ such that

$$(22) \quad \left| \frac{1}{|x - x_0|} \left| \int_{x_0}^x |f(x) - \alpha| dx \right| - |f(x) - \alpha| \right| < \frac{\epsilon}{3} \quad (|x - x_0| < \delta).$$

Combining (20), (21) and (22), we obtain

$$\left| \frac{1}{|x - x_0|} \left| \int_{x_0}^x |f(x) - \beta| dx \right| - |f(x) - \beta| \right| < \epsilon \quad (|x - x_0| < \delta).$$

Hence $|f(x) - \beta|$ is for $x = x_0$ the derivative of its indefinite integral. Since x_0 was any point not of E_1 and β was any irrational number, it follows that $|f(x) - \gamma|$, where γ is any number, is the derivative of its indefinite integral for every point of $(a \leq x \leq b)$ except a set of measure zero. Since for any point x_0 we may choose $\gamma = f(x_0)$, our lemma is proved.

LEBESGUE'S THEOREM. *The Fourier development of a function $f(x)$ that is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π , is summable (C1) almost everywhere to the value $f(x)$.*

Our theorem may be proved by showing that

$$(23) \quad \lim_{n \rightarrow \infty} \left[\frac{S_n(x)}{n} - f(x) \right] = 0,$$

where $S_n(x)$ is defined by (3), is true almost everywhere. In view of (4) and Lemma 1 the expression in brackets in (23)

may be written in the form

$$(24) \quad \frac{1}{n\pi} \int_0^{\pi/2} [f(x+2t) + f(x-2t) - 2f(x)] \frac{\sin^2 nt}{\sin^2 t} dt \\ = \frac{1}{n\pi} \int_0^{\pi/2} \varphi_x(t) \frac{\sin^2 nt}{\sin^2 t} dt.$$

We will show that (24) approaches zero as n becomes infinite for all values of x for which $|\varphi_x(t)|$ is for $t = 0$ the derivative of its indefinite integral $\Phi_x(t)$. This latter will be the case for all values of x for which both $|f(x+2t) - f(x)|$ and $|f(x-2t) - f(x)|$ are for $t = 0$ the derivatives of their indefinite integrals, and in view of Lemma 4 this is true almost everywhere. Thus our theorem will be proved.

We consider then a value of x for which $\Phi_x'(0) = \varphi_x(0) = 0$, and hence we have

$$(25) \quad \lim_{t \rightarrow 0} \left[\frac{\Phi_x(t)}{t} \right] = 0.$$

Integrating by parts in the integral on the right hand side of (24), this expression takes the form

$$(26) \quad \frac{1}{n\pi} \left[\Phi_x(t) \frac{\sin^2 nt}{\sin^2 t} \right]_0^{\pi/2} - \frac{1}{\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{\sin t} \cdot \frac{\sin 2nt}{\sin t} dt \\ + \frac{2}{n\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{\sin t} \cdot \frac{\sin^2 nt}{\sin^2 t} dt.$$

The first term in (26) vanishes at the lower limit in view of (25) and at the upper limit takes on a value which approaches zero as n becomes infinite. From Lemma 2 and (25) the third term is readily seen to approach zero as n becomes infinite. The second term may be replaced by

$$(27) \quad \frac{1}{\pi} \int_0^{\pi/2} \frac{\Phi_x(t)}{t} \cdot \frac{\sin 2nt}{\sin t} dt,$$

since the difference between the two approaches zero as n becomes infinite in view of the theorem of Riemann-Lebesgue referred to in the proof of Lemma 3. But it follows from Lemma 3 that (27) approaches zero as n becomes infinite. Hence (26), and therefore (24), has this same property for the value of x we are considering. Thus, as pointed out above, the theorem is proved.

We may also use for the summation of Fourier's series the non-integral orders of summability introduced by Knopp, Marcel Riesz and Chapman. These may be defined as follows:

Let

$$(28) \quad s_n = \sum_{m=0}^{m=n} u_m, \quad A_n^{(k)} = \frac{\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(n+1)},$$

$$S_n^{(k)} = \sum_{p=0}^{p=n} A_{n-p}^{(k-1)} s_p.$$

Then if $\sigma_n^{(k)} = S_n^{(k)}/A_n^{(k)}$ approaches a limit σ as n becomes infinite, we say that the series $\sum u_n$ is summable (Ck) with sum σ . We may also define the sum of the series to be

$$(29) \quad \lim_{\omega \rightarrow \infty} \left[\sum_{n=0}^{n \leq \omega} \left(1 - \frac{n}{\omega}\right)^k u_n \right]$$

whenever that limit exists. This latter definition has been shown by Riesz to be entirely equivalent to the former one.*

It was shown independently by Riesz [18] and Chapman [3] that the Fourier development of a function $f(x)$ having a Lebesgue integral is summable $(C, k > 0)$ at all points at which $\lim_{h \rightarrow 0} [\frac{1}{2}\{f(x+h) + f(x-h)\}]$ exists, to the value of that limit. It was shown by G. H. Hardy that the series is summable $(C, k > 0)$ to $f(x)$ almost everywhere [13]. Hardy's proof of his theorem is similar in method to a simplified proof of the Riesz-Chapman theorem given by W. H. Young [19] and [20]. Space is lacking to give here the details of these proofs. They depend on properties of the functions

$$C_p(t) = \frac{t^p}{\Gamma(p+1)} \left\{ 1 - \frac{t^2}{(p+1)(p+2)} + \frac{t^4}{(p+1)(p+2)(p+3)(p+4)} - \dots \right\}$$

introduced by Young. These functions are generalizations of the sine and cosine, for we have obviously $C_0(t) = \cos t$, $C_1(t) = \sin t$.

§2. Convergence Factors.

Convergence factors may be defined as a set of functions of a parameter which, when introduced as factors of the succes-

* Cf. *Comptes Rendus*, June 12, 1911.

sive terms of a series, cause a divergent series to converge,* or a series which is already convergent to converge more rapidly throughout a given range of values of the parameter. In the case of all the convergence factors used in practice, it is further true that each factor approaches unity as the parameter approaches a certain value, and that the function of the parameter defined by the series with the convergence factors approaches a limit as the parameter approaches this same value, this limit being the value of the series for convergent series and a value we find it useful to ascribe to the series in the case of a divergent series.

Thus we see that a set of convergence factors may be used to define the sum of a divergent series. This method of summation goes back to Euler, who frequently arrived at a value for a divergent series $\sum u_n$ by setting $\sum u_n = \lim_{x \rightarrow 1-0} \sum u_n x^n$. A simple illustration of this process is exhibited in the case of the series $1 - 1 + 1 - 1 + \dots$. For this series we have

$$\lim_{x \rightarrow 1-0} \sum u_n x^n = \lim_{x \rightarrow 1-0} (1 - x + x^2 - x^3 + \dots) = \lim_{x \rightarrow 1-0} \frac{1}{1+x} = \frac{1}{2}.$$

This value agrees with the value of the series when summed by the mean value process.

From one point of view, practically every method of summing divergent series may be regarded as a convergence factor method. Thus in the summation by first means we seek the limit as n becomes infinite of

$$F(n) = \frac{s_0 + s_1 + \dots + s_n}{n+1} = u_0 + \left(1 - \frac{1}{n+1}\right)u_1 + \left(1 - \frac{2}{n+1}\right)u_2 + \dots + \left(1 - \frac{n}{n+1}\right)u_n.$$

The expression on the right hand side may be regarded as the series obtained by introducing the convergence factors

$$f_m\left(\frac{1}{n}\right) = 1 - \frac{m}{n+1} \quad (m = 0, 1, 2, \dots, n);$$

$$f_m\left(\frac{1}{n}\right) = 0 \quad (m = n+1, n+2, \dots)$$

* Some writers have employed the term convergence factor in the case of a set of factors which cause a divergent series to tend toward convergence, i. e., to become summable of a lower index.

into the series Σu_n . Here both the variables of which the convergence factors are functions take on only discrete values and we take the limit of $F(n)$ as n becomes infinite. But the principal characteristic of this type of factors lies in the fact that for any definite value of the parameter they are all zero from a certain point on. In this they differ from the type to which the name convergence factor has ordinarily been applied.

The simplest example of the latter type is to be found in the set of convergence factors mentioned above as used by Euler, namely the set $1, x, x^2, \dots$. In connection with this set it is interesting to note that the fact we have indicated above with regard to the effect of their introduction into the series $1 - 1 + 1 - 1 + \dots$, is not an accidental coincidence but a particular case of a general theorem due to Frobenius [8]. This theorem may be stated as follows:

FROBENIUS'S THEOREM: *If the series Σu_n is summable (C1) to the value S , then the series $\Sigma u_n x^n$ will converge for $0 < x < 1$, and the function $F(x)$ which it defines for those values will be such that $\lim_{x \rightarrow 1-0} F(x) = S$.*

We shall not give the proof of this theorem, inasmuch as it is a special case of a more general theorem due to Bromwich [1]. This latter theorem includes also a number of other theorems about convergence factors due to various writers. In its most general form it applies to the introduction of convergence factors into series summable (Ck), where k is any positive integer.* We shall deal only with the simplest case where $k = 1$, but for this case we will state the theorem in somewhat more general form than it is given by Bromwich. It requires only slight changes of phraseology to fit the proof to this modified statement, and in this latter form the theorem applies directly to cases to which the other form is not applicable.

BROMWICH'S THEOREM: *If the series Σu_n is summable (C1) to the value S , and the set of functions, $f_0(\alpha), f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha), \dots$, defined for a set of values $E(\alpha)$ having at least one limit point α_0 , not of the set, satisfies the conditions*

* Bromwich's theorem has been generalized to cases where k is non-integral by Chapman and Ottolenghi. Cf. [3] and *Giornale di Matematiche di Battaglini*, vol. 49 (1911).

$$\left. \begin{aligned} (A) \quad & \sum_{n=p}^{n=q} n |\Delta^2 f_n(\alpha)|^* < K \quad \left(\begin{array}{l} p, q \text{ any integers} \\ K \text{ a positive constant} \end{array} \right) \\ (B) \quad & \lim_{n \rightarrow \infty} n f_n(\alpha) = 0 \\ (C) \quad & \lim_{\alpha \rightarrow \alpha_0} f_n(\alpha) = 1, \end{aligned} \right\} E(\alpha),$$

then the series $\sum u_n f_n(\alpha)$ will converge absolutely over $E(\alpha)$ and the function $F(\alpha)$ which it defines there will be such that $\lim_{\alpha \rightarrow \alpha_0} F(\alpha) = S.$ [†]

Since we have, using the notation (28) for the case $k = 1$,

$$\begin{aligned} u_n = s_n - s_{n-1} &= (S_n - S_{n-1}) - (S_{n-1} - S_{n-2}) \\ &= S_n - 2S_{n-1} + S_{n-2}, \end{aligned}$$

we obtain, if we set $S_{-2} = S_{-1} = 0$,

$$\begin{aligned} (30) \quad \sum_{m=0}^{m=n} u_m f_m(\alpha) &= \sum_{m=0}^{m=n} (S_m - 2S_{m-1} + S_{m-2}) f_m(\alpha) \\ &= \sum_{m=0}^{m=n-1} S_m \Delta^2 f_m(\alpha) + S_n f_n(\alpha) - S_{n-1} f_{n+1}(\alpha). \end{aligned}$$

In view of the hypothesis that $\sum u_n$ is summable (C1) we may choose a positive constant C such that $|S_n| < (n+1)C$ for all values of n . Then, making use of this fact and condition (A), we see that the summation on the right hand side of (30) approaches a limit over $E(\alpha)$ as n becomes infinite. Making use of the property of S_n just employed and condition (B), it follows that the remaining two terms on the right hand side of (30) approach zero as n becomes infinite. Thus the left hand side of (30) approaches a limit also, and we have

$$(31) \quad F(\alpha) = \sum_{m=0}^{\infty} u_m f_m(\alpha) = \sum_{m=0}^{\infty} S_m \Delta^2 f_m(\alpha).$$

Applying this identity to the series $1 + 0 + 0 + 0 + \dots$, we obtain, since in this case $S_m = m + 1$,

$$(32) \quad f_0(\alpha) = \sum_{m=0}^{\infty} (m+1) \Delta^2 f_m(\alpha).$$

* The notation $\Delta^2 f_n(\alpha)$ is used as an abbreviation for $f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)$.

† If the terms u_n are functions of a variable and $\sum u_n$ is uniformly summable throughout a certain interval, the limit of $F(\alpha)$ will be approached uniformly over that interval as α approaches α_0 .

From (31) and (32) we obtain

$$(33) \quad F(\alpha) - Sf_0(\alpha) = \sum_{m=0}^{\infty} \left(\frac{S_m}{m+1} - S \right) (m+1) \Delta^2 f_m(\alpha).$$

Since Σu_n is summable (C1) to S , we may choose an r such that

$$(34) \quad \left| \frac{S_m}{m+1} - S \right| < \frac{\epsilon}{4K} \quad (m \geq r),$$

ϵ being an arbitrary positive quantity and K the K of condition (A). Then, r being fixed, we may in view of condition (C) choose δ such that

$$(35) \quad \left| \sum_{m=0}^{m=r-1} \left(\frac{S_m}{m+1} - S \right) (m+1) \Delta^2 f_m(\alpha) \right| < \frac{\epsilon}{2} \quad (|\alpha - \alpha_0| < \delta).$$

From (33), (34), condition (A) and (35) we obtain

$$|F(\alpha) - Sf_0(\alpha)| < \epsilon \quad (|\alpha - \alpha_0| < \delta).$$

Whence

$$\lim_{\alpha \rightarrow \alpha_0} F(\alpha) = \lim_{\alpha \rightarrow \alpha_0} Sf_0(\alpha) = S,$$

and our theorem is proved.

§3. *Applications to Problems in the Flow of Heat.*

The theorems about convergence factors are of special interest in view of the fact that they have important applications in connection with certain problems in mathematical physics. We will illustrate this by discussing a particular problem in the flow of heat.

We take the case of a finite rod of length π whose ends are maintained at zero temperature and whose surface is a non-conductor. Suppose the cross section of the bar is so small that the temperature is sensibly constant throughout it, and suppose the initial temperature is given by the function $f(x)$. We wish to determine the temperature of any point of the bar at any later moment.

Our problem reduces to the determination of a function of x and t , $v(x, t)$, such that

$$(\alpha) \quad \frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \quad \left(\begin{array}{l} 0 < x < \pi \\ t > 0 \end{array} \right),$$

$$(\beta) \quad v(0, t) = 0 = v(\pi, t) \quad (t > 0),$$

$$(\gamma) \quad \lim_{\substack{t \rightarrow +0 \\ x \rightarrow x_1}} v(x, t) = f(x_1),$$

where x_1 is any point in whose neighborhood $f(x)$ is continuous. If we assume that $v(x, t)$ may be expressed in the form $u(x)w(t)$, we find as particular solutions of (α)

$$(36) \quad Ae^{-ka^2t} \sin ax, \quad Be^{-ka^2t} \cos ax \quad (A, B \text{ and } a \text{ constants}).$$

The second solution does not satisfy the boundary condition (β) , so we reject it. The first solution satisfies (β) , but will satisfy (γ) only in case $f(x) = \sin ax$. If $f(x)$ does not have this form, we naturally try to build up a sum of particular solutions of (α) of the form of the first solution in (36), $\sum A_n e^{-ka_n^2t} \sin a_n x$, which will approach $f(x)$ as t approaches $+0$. This raises the question of the possibility of expressing $f(x)$ in the form $\sum A_n \sin a_n x$. We know from the theory of convergent Fourier series that an arbitrary function of x , satisfying fairly wide conditions, may be expressed in a series of this form. But we also know that there exist continuous functions of x whose Fourier development diverges at points everywhere dense.

Our physical intuition tells us that there must be a solution of our problem corresponding to any original distribution of temperature that is thinkable, and therefore certainly in the case of an original distribution that is continuous. Fejér's theorem about summability (C1) of the Fourier series, combined with some general theorem about convergence factors such as Bromwich's theorem, furnishes the mathematical demonstration that the series formed by introducing convergence factors of the type e^{-kn^2t} into the sine series for $f(x)$ is the desired solution.

The proof in detail is relatively brief. We have to show that the series

$$(37) \quad \sum_{n=1}^{\infty} A_n \sin nx e^{-kn^2t} \quad \left(A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \right)$$

converges for $(t > 0; 0 \leq x \leq \pi)$, and defines in that region a function $v(x, t)$ which satisfies conditions (α) , (β) and (γ) . We know from the Riemann-Lebesgue theorem referred to above that A_n approaches zero as a limit as n becomes infinite, and therefore the series (37) and the various derived

series obtained by differentiating (37) term by term with regard to t or x , will converge uniformly throughout the region ($t \geq t_0 > 0, 0 \leq x \leq \pi$). Thus we see that the series (37) defines a function $v(x, t)$ that satisfies conditions (α) and (β). It remains to show that this function also satisfies (γ).

It is readily seen that $v(x, t)$ will satisfy (γ) provided it approaches $f(x)$ as a limit as t approaches $+0$, uniformly throughout any interval included in an interval in which $f(x)$ is continuous. Since, from Fejér's theorem, $\sum A_n \sin nx$ is uniformly summable throughout any interval included in an interval of continuity of $f(x)$, it will follow from Bromwich's theorem that $v(x, t)$ approaches $f(x)$ uniformly throughout such an interval, in case the convergence factors in (37) satisfy the conditions of that theorem.

That conditions (B) and (C) are satisfied is easily seen. Turning to condition (A), we find that $\Delta^2 e^{-kn^2t}$ is negative when $n \leq \sqrt{1/2kt} - 2$ and is positive when $n \geq \sqrt{1/2kt}$. Hence this expression cannot change sign more than three times for any value of $t > 0$, and therefore condition (A) will be satisfied if we can determine a positive constant such that any sequence of terms chosen from $\sum n \Delta^2 e^{-kn^2t}$ is less in absolute value than this constant for all values of $t > 0$.

We have

$$\begin{aligned} \sum_{n=p}^{n=q} n \Delta^2 e^{-kn^2t} &= p e^{-kp^2t} - (p-1) e^{-k(p+1)^2t} \\ &\quad - (q+1) e^{-k(q+1)^2t} + q e^{-k(q+2)^2t} \\ &= p [e^{-kp^2t} - e^{-k(p+1)^2t}] - q [e^{-k(q+1)^2t} \\ &\quad - e^{-k(q+2)^2t}] + e^{-k(p+1)^2t} - e^{-k(q+1)^2t} \\ &= 2kp(p+\theta_1) t e^{-k(p+\theta_1)^2t} - 2kq(q+\theta_2) t e^{-k(q+\theta_2)^2t} \\ &\quad + e^{-k(p+1)^2t} - e^{-k(q+1)^2t} \quad \left(\begin{array}{l} 0 < \theta_1 < 1 \\ 1 < \theta_2 < 2 \end{array} \right). \end{aligned}$$

Each of the first two terms on the right hand side is less in absolute value than the expression $2kye^{-y}$ for some value of $y > 0$, and since this expression and also e^{-y} remain finite for all values of $y > 0$, it follows that condition (A) is satisfied by the convergence factors of (37). Hence the series (37) defines a function $v(x, t)$ which satisfies condition (γ), and as we saw previously that this same function satisfies (α) and (β), it follows that our physical problem is completely solved.

§4. *Summability of Other Developments.*

The developments in orthogonal functions of one variable that have been most extensively studied are, aside from the Fourier series, the developments in Sturm-Liouville functions and in Legendre's and Bessel's functions. The Sturm-Liouville developments offer less difficulty and yield simpler results than the other two. This is due to the fact that the differential equations which define Legendre's and Bessel's functions have singular points in the interval in which we wish to develop an arbitrary function, while the differential equation for the Sturm-Liouville functions does not. It is in the neighborhood of these singular points that the developments in terms of the former functions are more complex in their behavior and are more difficult to handle.

Very complete results with regard to the summability (C1) of the Sturm-Liouville developments were obtained by Haar in his dissertation [11]. He showed in fact that the behavior of the development of any function having a Lebesgue integral was the same at any point as the behavior of the cosine development of the same function. This result follows readily from the following fundamental theorem:

HAAR'S THEOREM. *If we represent by $s_n(x)$ and $\sigma_n(x)$ the sums of the first $n + 1$ terms of the Sturm-Liouville and cosine developments respectively of any function $f(x)$ having a Lebesgue integral, we have*

$$\lim_{n \rightarrow \infty} [s_n(x) - \sigma_n(x)] = 0$$

*uniformly over the interval $(0 \leq x \leq \pi)$.**

This theorem enables us to infer from the various results obtained with regard to the summability of the Fourier's series, corresponding results for the Sturm-Liouville developments. Thus we obtain not only the analogue of Fejér's theorem, as pointed out by Haar, but also the analogues of Lebesgue's theorem, the Riesz-Chapman theorem, and Hardy's theorem.

The first study of the summability of the developments in Legendre's functions was made in 1908 by Fejér who established summability (H2), or what is equivalent (C2), at every

* If the interval of definition of the Sturm-Liouville functions is taken as some interval (a, b) differing from the interval $(0, \pi)$, we readily reduce that case to the above by a change of variable in the differential equation and boundary conditions by means of which the functions are defined.

point in the interval $(-1 \leq x \leq 1)$ at which the function is continuous or has a finite jump, provided the function is absolutely integrable [6]. In 1911 Haar proved the summability $(C1)$ of the development at interior points of the interval $(-1 \leq x \leq 1)$ for the case where the function developed is continuous throughout the whole interval [12]. During the same year Chapman independently established summability $(C, k > 1)$ for all points of the interval at which the function is continuous or has a finite jump, provided the function has a Lebesgue integral over the interval. He further obtained summability $(C, k > 1/2)$ at the end points and $(C, k > 0)$ at interior points for the case where the function satisfies certain additional restrictions as to the possession of limited variation [4]. In 1913 Gronwall obtained summability of the same orders with less restriction on the function developed. His requirement for the case of interior points is that $f(x)$ and $(1 - x^2)^{(k/2) - (1/4)}f(x)$ should be absolutely integrable in $(-1, 1)$. For the end points he requires the absolute integrability of $f(x)$ and a further condition on the function in the neighborhood of the end point opposite to the one for which summability is established. This condition, in the case of summability at $+1$, demands that

$$\int_{-1}^{-1+\delta} f(x') \sigma_n((2n+1)P_n(x)P_n(x')) dx',$$

where $P_n(x)$ represents the Legendrian of the n th order and $\sigma_n^{(k)}(u_n) = S_n^{(k)}/A_n^{(k)}$, $S_n^{(k)}$ and $A_n^{(k)}$ being defined by equation (28), should, for some value of $\delta > 0$, approach zero as n becomes infinite [10].

In the case of the developments in Bessel's functions, summability $(C1)$ at points in the interval $(0 < x < 1)$ where the function developed is continuous or has a finite jump, was established by the writer in 1908 [15] for the case of a function that is finite and integrable. Summability $(C1)$ for points of the same nature in $(0 < x \leq 1)$, was established by W. B. Ford in his recent book on divergent series and summability [7] for the case of a function $f(x)$ that becomes infinite at a finite number of points while $\sqrt{x}f(x)$ remains absolutely integrable. Summability $(C, k > 0)$ at points in the interval $(0 < x \leq 1)$ at which the function is continuous or has a finite jump for a function $f(x)$ such that $\sqrt{x}f(x)$ has a Lebesgue

integral, and summability ($C\frac{1}{2}$) for $x = 0$, provided $f(x)$ has a Lebesgue integral, is continuous at the origin and is such that $|(f(x) - f(0))/x^\gamma|$ is less than a positive constant for some value of $\gamma > \frac{1}{2}$ and values of x in the neighborhood of zero, have been established by the writer in two papers that have been presented to the Society but not as yet published.* Thus we see that here as in the case of the developments in Legendre's functions, we have to put more restriction on the function developed at the point at which the differential equation corresponding to the functions in terms of which we develop has a singular point; moreover, even then we do not get summability of as low an order. The analogy between the two cases suggests the existence of a general theory† with regard to the summability of developments in orthogonal functions that satisfy linear differential equations with singular points, which will include the important features of both cases. As far as the writer is aware this general theory is yet to be developed.

§5. *The Summability of Double Series.*

All the different methods of summation used in connection with simple series can be readily generalized so as to furnish methods for summing double series. We shall discuss here only the extension of the simplest method, i. e., summation by arithmetic means of the sums, to double series. We consider the double series Σa_{ij} , and we set

$$s_{mn} = \sum_{i=0, j=0}^{i=m, j=n} a_{ij}, \quad S_{mn} = \sum_{i=0, j=0}^{i=m, j=n} s_{ij},$$

$$\sigma_{mn} = S_{mn}/(m+1)(n+1).$$

Then if $\lim_{m, n \rightarrow \infty} \sigma_{mn}$ exists and is equal to σ , we say that the series Σa_{ij} is summable ($C1$) to the value σ . It is easy to establish the consistency of this definition with the Pringsheim definition of convergence, for series such that $|s_{mn}| < C$, a positive constant, for all values of m and n . Moreover, in that case it will also be true that $|\sigma_{mn}| < C$ for all values of m and n .

The theorem of Frobenius for simple series has been extended to double series by Bromwich and Hardy [2], and Brom-

* For abstracts of these papers, see BULLETIN, vol. 24, pp. 65 and 422.

† Cf. a principle of generalization laid down by E. H. Moore, Introduction to a Form of General Analysis, p. 1.

wich's theorem for simple series has been extended to double series by the writer [16]. Fejér's theorem about the summability of the ordinary Fourier series has been extended to the double Fourier series, for points of continuity* of the function developed, by W. H. Young [21] and the writer [16], and for points of discontinuity of certain types by the writer [17]. These extensions, taken in connection with the extension of the theorem on convergence factors, enable us to discuss certain problems in mathematical physics in which double Fourier series occur, in a manner analogous to the discussion of the problem in the flow of heat considered earlier in the paper. Such a discussion may be found in [16].

The consideration of the summability of the double Fourier series naturally suggests the consideration of the summability of double series involving other orthogonal functions. Furthermore, the extension of the conception of summability to double series leads naturally to its further extension to triple series and to multiple series of any order. In particular we might study the summability of the triple Fourier series and thus obtain results having important applications to mathematical physics, analogous to those mentioned in connection with the ordinary and double Fourier series. Many other special studies readily suggest themselves. Thus we see that the work already done on the summability of developments in orthogonal functions is only a beginning, and that a large unexplored field remains to be investigated.

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* This same extension was also made independently by Küstermann. Cf. his dissertation, "Ueber Fouriersche Doppelreihen und das Poissonsche Doppelintegral," Munich (1913).

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MODULAR SYSTEMS.

The Algebraic Theory of Modular Systems. By F. S. MACAULAY. [Cambridge Tracts in Mathematics and Mathematical Physics, No. 19.] Cambridge University Press, 1916. xiv + 112 pp.

A MODULAR system is an infinite aggregate of polynomials in n variables $x_1, x_2, \dots, x_n$, defined by the property that if F, F_1, F_2 belong to the system, $F_1 + F_2$ and AF also belong to the system, where A is any polynomial in $x_1, x_2, \dots, x_n$. Hence if $F_1, F_2, \dots, F_k$ belong to a modular system so also does $A_1F_1 + A_2F_2 + \dots + A_kF_k$, where $A_1, A_2, \dots, A_k$ are arbitrary polynomials in $x_1, x_2, \dots, x_n$. In the algebraic theory (to which this tract is devoted) polynomials such as F and aF , where a is a quantity not involving the variables, are regarded as the same polynomial.

Let us consider any given infinite aggregate of polynomials in $x_1, x_2, \dots, x_n$. Of those of any given degree L there are only a finite number which are linearly independent. Hence we may select an ordered sequence of polynomials $F_1, F_2, F_3, \dots$ of the given aggregate such that any given polynomial of the aggregate is linearly expressible in terms of a finite number of the polynomials $F_1, F_2, F_3, \dots$. Concerning such an ordered enumerable sequence of polynomials Hilbert has established the following theorem:

If $F_1, F_2, F_3, \dots$ is an infinite sequence of polynomials in n variables $x_1, x_2, \dots, x_n$, then there exists a finite number k such that for $h > k$ we have a relation of the form

$$F_h = A_{h1}F_1 + A_{h2}F_2 + \dots + A_{hk}F_k$$

in which the A_{ij} denote polynomials in $x_1, x_2, \dots, x_n$.

Our author gives on page 38 essentially König's proof of this theorem. Since the proof is from first principles, he is justified in assuming the theorem from the beginning.

From the foregoing considerations it follows that every infinite aggregate of polynomials is contained in the infinite aggregate of a modular system and that every modular system is itself the aggregate of polynomials

$$A_1F_1 + A_2F_2 + \dots + A_kF_k$$

in which $F_1, F_2, \dots, F_k$ are k properly chosen polynomials of the system and $A_1, A_2, \dots, A_k$ are arbitrary polynomials. The polynomials $F_1, F_2, \dots, F_k$ are said to form a basis of the system.

In its simpler aspects the theory of modular systems is of importance in geometry. Let it be required, for example, to determine the class of algebraic plane curves which pass through the intersection points of the curves

$$F_1(x, y) = 0, \quad F_2(x, y) = 0.$$

It is obvious that the class includes all the curves

$$A_1F_1 + A_2F_2 = 0$$

where A_1 and A_2 are polynomials in x and y . The question arises as to whether or under what circumstances every algebraic curve passing through the given intersection points can be represented by an equation of the last form above. The

answer to such a question affords one of the simplest applications of the algebraic theory of modular systems.

The primary object of the algebraic theory of modular systems is to discover those general properties of a system which will afford a means of answering the question whether a given polynomial is or is not a member of a given system. This calls for a generalization of the theory of the solution of equations in several unknowns. In order that a polynomial F may belong to a modular system with a given basis $F_1, F_2, \dots, F_k$ it is obviously necessary that F shall vanish for all finite solutions of the system $F_1 = F_2 = \dots = F_k = 0$. This condition is sufficient only if the given modular system has a certain property; otherwise, it is not sufficient, and F must satisfy further conditions, also connected with the solutions of the system $F_1 = F_2 = \dots = F_k = 0$; and these may be difficult to express concretely.

At present the theory of modular systems is incomplete and offers a wide field for research. The subject is one of great difficulty, and he who works in it must exercise extreme care to avoid the pitfalls. Macaulay points out several mistakes made by his predecessors, some of them occurring in the most important memoirs dealing with the subject.

From what has been said it is evident that the first step in the theory of modular systems is to find all the solutions of the algebraic equations $F_1 = F_2 = \dots = F_k = 0$. This is completely accomplished in the theories of the resultant and the resolvent, developed by our author in sections I and II (pages 3–28). The theory of the resultant of two homogeneous polynomials in two variables is first treated. There is given then a parallel development of the much more difficult theory of the resultant of n homogeneous polynomials in n variables. From this the necessary theory for non-homogeneous polynomials follows at once. Application is made to the solution of certain classes of algebraic equations. In the theory of the resolvent the author follows in the main König's exposition of Kronecker's method of solving equations by means of the resolvent.

General properties of modular systems are developed in section III (pages 29–63). The treatment is closely allied to Lasker's memoir (*Mathematische Annalen*, volume 60, 1905) and Dedekind's theory of ideals. Section IV (pages 64–100) contains an extension of Lasker's results founded on the methods

originated by Noether. The contents of this chapter are due to the author himself. A considerable number of the properties proved in the section have been established by him in previous memoirs. But a new method, that of the inverse system, is here employed for the first time and the results are closely associated with it. The author's own account of the method is to be found on pages 64 ff.

The monograph ends with a note of twelve closely printed pages containing a brief explanation of the theory of ideals of algebraic numbers and functions and of the relation in which the algebraic theory of modular systems stands with respect to it.

Throughout the tract the exposition is given in condensed form, evidently best adapted to the needs of investigators in the field. But a portion of the treatment, especially that of the first two sections, is suited to the needs of the general mathematical reader interested in the general aspects of the theory of algebraic equations in several unknown quantities.

R. D. CARMICHAEL.

NOTES.

THE regular meeting of the Chicago Section of the American Mathematical Society at the University of Chicago on Friday and Saturday, April 4-5, 1919, will include a symposium on the geometry of numbers with applications to questions of minima and algebraic numbers. Formal papers, based largely on the work of Minkowski, will be presented by Professor H. F. BLICHFELDT, of Stanford University, and Professor L. E. DICKSON, of the University of Chicago. Synopses of these papers will be sent out with the programmes of the meeting.

THE programme of the regular meeting of the Society in New York City on April 26, 1919, will include reports of the work of members of the Society in the government Ordnance Department at Washington and Aberdeen.

THE opening (January) number of volume 20 of the *Transactions of the American Mathematical Society* contains the following papers: "Necessary conditions in the problems of

Mayer in the calculus of variations," by GILLIE A. LAREW; "Linear equations with unsymmetric systems of coefficients," by ANNA J. PELL; "On convex functions," by HENRY BLUMBERG; "Projective transformations in function space," by L. L. DINES; "On the order of primitive groups (IV)," by W. A. MANNING.

There is a portrait of the late Professor MAXIME BÔCHER as frontispiece, with a short obituary note. Copies of this portrait may be obtained by sending twenty cents in postage stamps to the office of the Society, 501 West 116th Street, New York City.

THE opening (January) number of volume 41 of the *American Journal of Mathematics* contains the following papers: "Groups generated by two operators whose relative transforms are equal to each other," by G. A. MILLER; "A classification of general (2, 3) point correspondences between two planes," by T. R. HOLLCROFT; "The classification of plane involutions of order (3)," by ANNA M. HOWE; "On surfaces containing a system of cubics that do not constitute a pencil," by C. H. SISAM; "An isoperimetric problem with variable end-points," by A. S. MERRILL.

WITH the January, 1919, issue, the *Tôkyô Sûgaku-Buturigakkwai Kizi* (*Proceedings of the Tokyo mathematico-physical Society*) begins its third series. The name of the periodical changes with the new series to *Nippon Sûgaku-Buturigakkwai Kizi* (*Proceedings of the physico-mathematical Society of Japan*).

THE January number of the *American Mathematical Monthly* contains the names and rank of one hundred and seventy-eight teachers of mathematics in the national war service.

AT the recent annual meeting of the Mathematical Association of America Professor H. E. SLAUGHT was elected president and Professors R. G. D. RICHARDSON and H. L. RIETZ vice-presidents.

AT the meeting of the London mathematical society held December 12, the following papers were read: By G. H. HARDY and J. E. LITTLEWOOD, "Applications of the method of Faray dissection in the analytic theory of numbers: (1) A new solu-

tion of Waring's problem; (2) Proof that every large number is the sum of at most thirty-three biquadrates; (3) The Riemann hypothesis and the expression of a number as the sum of a stated number of primes"; by N. M. SHAH and B. M. WILSON, "Numerical data connected with Goldbach's theorem"; by M. FRÉCHET, "Integrals in abstract fields."

AT the meeting of the Edinburgh mathematical society on January 10, the following papers were read: By E. M. HORSBURGH, "A mechanical solution of a differential equation of torsion"; by F. BOWMAN, "Note on the intersection of a plane curve and its Hessian at a multiple point" and "Note on the formula for the radius of the circumsphere of a tetrahedron in terms of the edges."

THE Prince Jablonowski Society of Leipzig announces the following prize problem for 1921: "To extend the theory of linear functional differential equations in any direction. A complete treatment of new special cases is especially desirable." Competing memoirs should be submitted not later than October 31, 1920. The value of the prize is 1500 marks.

THE philosophical faculty of the University of Berlin announces the following prize problem: "To determine, by means of the theory of elementary divisors, the criteria that a given matrix be capable of representation as the composition of two skew-symmetric matrices." Competing memoirs should be presented before June 4, 1919.

THE academy of sciences of Heidelberg has joined the other learned societies as contributing member to the support of the *Encyklopädie der mathematischen Wissenschaften*.

COLLÈGE DE FRANCE. THE following courses in mathematics are announced for the session beginning December 2, 1918: By Professor G. HUMBERT: Theory of quadratic numbers, two hours.—By Professor J. HADAMARD: Influence of the form of the domain in the problems of mathematical physics, two hours.—By Professor M. BRILLOUIN: English and American theories of the gravitational stability of the earth, two hours.—By Professor L. LANGEVIN: The principle of relativity and the theories of gravitation, two hours.—By Pro-

fessor LE ROY: The present state of mathematical philosophy, and its relations with the philosophy of intuition, two hours.

PROFESSOR M. FRÉCHET, of the Sorbonne, has been appointed professor of mathematics in the reconstituted University of Strassbourg. During the summer semester of the present year (April 15–July 30) he will lecture on General Analysis (Calcul fonctionnel). The lectures will be in French, but Professor Fréchet offers to discuss them in English with any English-speaking students who wish to avail themselves of the opportunity.

THE University of Frankfort has conferred an honorary doctorate on Professor L. KÖNIGSBERGER, of the University of Heidelberg.

DR. V. GEILEN has been appointed docent in mathematics at the University of Münster.

PROFESSOR OTTO STAUDE, of the University of Rostock, has been chosen rector for the year 1918–19.

DR. V. GRILLIER, of the University of Biel, has been appointed professor of mathematics at the University of Bern.

• PROFESSOR M. NOETHER, of the University of Erlangen, has retired from active teaching, after forty-six years of service.

THE Royal Society of London has awarded its Copley medal to Professor H. A. LORENTZ, of the University of Leyden, for his work in mathematical physics.

PROFESSOR W. FOORD-KELCEY, of the Royal military academy, has been appointed Officer of the Order of the British Empire, for services in connection with the war.

THE Rumford Committee of the American Academy of Arts and Sciences has voted the sum of \$500.00 to Professor A. G. WEBSTER, of Clark University, in aid of his researches in pyrodynamics and practical interior ballistics.

THE committee recently organized to establish a suitable memorial of the late Professor MAXIME BÔCHER requests that

subscriptions to the memorial fund be sent to the treasurer of the committee, Professor J. H. TANNER, Cornell University, Ithaca, N. Y. Contributors are invited to express their views and wishes in regard to the use which should be made of the fund.

At the University of Manitoba, Dr. H. R. KINGSTON has been promoted to an assistant professorship of mathematics. Assistant professor L. A. H. WARREN has been appointed acting professor of mathematics and astronomy, in the absence of Professor N. B. MACLEAN, who is major in the Canadian artillery service in France.

PROFESSOR E. R. HEDRICK, of the University of Missouri, has recently sailed for France, where he is to take charge of the mathematical work for American soldiers in the Y. M. C. A. system.

DR. L. R. FORD and Mr. R. S. TUCKER have been appointed instructors in mathematics at Harvard University.

At Cornell University, Mr. P. A. FRALEIGH, who was stationed at Aberdeen proving ground, has returned to his instructorship in mathematics. Mr. H. L. SMITH, who was under Major F. R. MOULTON in Washington, and Mrs. HELEN B. OWENS have been appointed instructors in mathematics.

MR. L. E. ARMSTRONG, of Stevens Institute of Technology, has been promoted to an assistant professorship of mathematics.

MR. J. W. BALDWIN, of Michigan State Normal College, has been appointed instructor in mathematics in Detroit Junior College.

PROFESSOR D. A. ROTHROCK, of Indiana University, was recently elected a member of the State Legislature. He is relieved of duties at the University during the legislative session to begin in January.

MR. F. S. NOWLAN, of Bowdoin College, has been promoted to an assistant professorship of mathematics.

AT Brown University, Mr. T. A. CORNELL and Mr. W. R. BURWELL have been appointed instructors and Dr. A. B. FRIZELL lecturer in mathematics. Mr. C. R. ADAMS has resigned his instructorship to accept the Grand Army fellowship.

DR. TOBIAS DANTZIG and Dr. G. A. PFEIFFER have been appointed instructors in mathematics at Columbia University.

AT the University of Minnesota, Dr. C. H. YEATON has been appointed instructor in mathematics. Professor G. N. BAUER is on leave of absence, and Professor H. L. SLOBIN has resigned.

MR. R. M. MATHEWS, of the Junior College, Riverside, Cal., has resigned to enter secondary teaching.

MR. ARTHUR RAMSEY, of Grove City College, has been promoted to an assistant professorship of mathematics.

AT Northwestern University, the following new instructors in mathematics have been appointed: Miss JESSICA M. YOUNG, Mr. T. DOLL, Dr. M. G. SMITH, and Mr. P. E. HEMKE.

AT Wesleyan University, Mr. C. L. STEARNS has been appointed instructor in mathematics and assistant in the Van Vleck observatory.

PROFESSOR G. MILHAUD, of the University of Paris, well known for his writings on the history and philosophy of Greek mathematics, died October 1, 1918, at the age of sixty years.

PROFESSOR F. DANIELS, of the University of Fribourg, Switzerland, died November 16, 1918, at the age of fifty-eight years.

PROFESSOR ULISSE DINI, of the R. Scuole Normale Superiore of Pisa, and editor of the *Annali di Matematica*, died October 28, 1918, at the age of seventy-three years.

PROFESSOR L. M. SYLOW, of the University of Christiania, died September 7, 1918, at the age of eighty-five years.

PROFESSOR E. LAMPE, of the technical school at Berlin, died September 4, 1918, at the age of seventy-eight years. He was an editor of the *Jahrbuch über die Fortschritte der Mathematik* and of the *Archiv der Mathematik und Physik*.

PROFESSOR P. VON SCHAEWEN died April 28, 1918, at the age of seventy-one years.

The death is announced of Commander A. N. SKINNER, U. S. N., retired, professor of mathematics at the U. S. Naval Academy from 1898 to 1907.

DR. PAUL CARUS, for many years editor of the *Monist* and the *Open Court*, died at La Salle, Ill., on February 11, 1919.

DR. G. M. GREEN, of Harvard University, author of numerous memoirs in projective differential geometry, died January 24, 1919, at the age of twenty-seven years. He had been a member of the American Mathematical Society since 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIEBERBACH (L.). Differentialrechnung. Leitfäden für den mathematischen und technischen Hochschulunterricht. Leipzig, Teubner, 1917. M. 2.80

BRENDEL (M.). See GALLE (A.).

CARATHÉODORY (C.). Vorlesungen über reelle Funktionen. Leipzig, Teubner, 1918. M. 30.00

DIECK (W.). Nichteuklidische Geometrie in der Kugelebene. (Mathematisch-physikalische Bibliothek, Band 31.) Leipzig, Teubner, 1918. M. 1.00

FRICK (H.). Ueber den Zusammenhang der Perioden quadratischer Formen positiver Determinante mit der Zerlegung einer Zahl in die Summe zweier Quadrate. (Diss., Eidgenöss. Technische Hochschule.) Zürich, 1918.

GALLE (A.) und STÄCKEL (P.). Materialien für eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein, M. Brendel, und L. Schlesinger. Heft 4: C. F. Gauss als Zahlenrechner von A. Galle; Heft 5: C. F. Gauss als Geometer von P. Stäckel. Leipzig, Teubner, 1918. Gr. 8vo. 142 pp. M. 5.60

KLATT (—.). See SCHMIEDEBERG (—.).

KLEIN (F.). See GALLE (A.).

- LEVI (F.). Abelsche Gruppen mit abzählbaren Elementen. Habilitationsschrift. Leipzig, Teubner, 1918.
- LUCKEY (P.). Einführung in die Nomographie. 1ter Teil: Die Funktionsleiter. (Mathematisch-physikalische Bibliothek, Band 28.) Leipzig, Teubner, 1918. M. 1.00
- D'OVIDIO (E.). See SCRITTI MATEMATICI.
- SCHLESINGER (L.). See GALLE (A.).
- SCHMIEDEBERG (—.), WETZSTEIN (—.) und KLATT (—.). Die Bedeutung des mathematischen und naturwissenschaftlichen Unterrichts für die Erziehung unserer Jugend. Preisschriften des Vereins zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts. Berlin, Salle, 1917. M. 4.50
- SCHNEIDER (R.). Die Beweisführung für die Richtigkeit des Fermatschen Satzes. Eine mathematische Studie. 2 Bände. Budapest, 1918. M.10.00 + 10.00
- SCRITTI MATEMATICI offerti ad Enrico d'Ovidio in occasione del suo LXXV genetliaco 11 agosto 1918. Torino, Fratelli Bocca, 1918. 8vo. 16 + 386 pp. L. 30.00
- SIMON (M.). Analytische Geometrie der Ebene. 3te Auflage. Berlin, 1918. M.1.00
- SLICHTER (C. S.). Elementary mathematical analysis. 2d edition. New York, McGraw-Hill, 1918. 497 pp. \$2.50
- STÄCKEL (P.). See GALLE (A.).
- SÜNDERHAUF (—.). Einführung in die höhere Mathematik. 2tes Heft: Determinanten. (Beihefte zur Zeitschrift Lehrerfortbildung.) Leipzig, Haase, 1918. 64 pp. M. 2.00
- VEBLEN (O.) and YOUNG (J. W.). Projective geometry. Volume 2, by O. Veblen. Boston, Ginn, 1918. 8vo. 12 + 511 pp. \$5.00
- WETZSTEIN (—.). See SCHMIEDEBERG (—.).
- WEYL (H.). Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis. Leipzig, 1918. M. 4.00
- YOUNG (J. W.). See VEBLEN (O.).

II. ELEMENTARY MATHEMATICS.

- ADLER (A.). Fünfstellige Logarithmen. Mit mehreren graphischen Rechentafern und häufig vorkommenden Zahlwerten. Berlin, 1918. M. 1.00
- BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Unterstufe. Ausgabe B. Leipzig, Teubner, 1917. 328 pp. Geb. M. 3.60
- CRANTZ (P.). Planimetrie. 2te Auflage. (Aus Natur und Geisteswelt, Band 340.) Leipzig, 1918. 117 pp. M. 1.50
- GLASER (R.). Sammlung von Aufgaben aus der Stereometrie. (Sammlung Göschen, Nr. 779.) Leipzig, Göschen, 1917. 168 pp.
- GÖTTING (E.). See BEHRENDSEN (O.).

- JACOB (J.), SCHIFFNER (F.), und TRAVNIČEK (J.). Lehrbuch der Arithmetik und Geometrie für Gymnasien und Realgymnasien. Wien, Deuticke, 1917-1918. Arithmetik, von Jacob, 1ter Teil: Unterstufe, 4te Auflage, 1917, 156 pp.; 2ter Teil, Mittelstufe, 2te Auflage, 1917, 146 pp.; Schlüssel: Die Theorie in Frageform und die Resultate der Aufgaben, 2ter Teil, 1917, 82 pp. Geometrie von Schiffner und Travniček, Mittelstufe, 2te Auflage, 1918, 221 pp. Geb.
Kr. 3.20 + 3.00 + 3.60 + 4.00
- MAENNCHEN (P.). Geheimnisse der Rechenkünstler. 2te Auflage. (Mathematisch-physikalische Bibliothek, Band 13.) Leipzig, Teubner, 1918. 50 pp. M. 1.00
- PFAU (J.). Erste Einführung in die Dreiecksrechenlehre (Trigonometrie) auf Grund der natürlichen Winkelfolgen zum Schul- und Selbstunterricht. Wien, Pichler, 1917. Geh. Kr. 3.60
- SCHIFFNER (F.). See JACOB (J.).
- SPORER (B.). Niedere Analysis. 2te Auflage. Berlin, 1918. M. 1.00
- TIMERDING (H. E.). Der goldene Schnitt. (Mathematisch-physikalische Bibliothek, Band 32.) Leipzig, Teubner, 1918. M. 1.00
- TRAVNIČEK (J.). See JACOB (J.).
- WUNSCH (H.). Unterhaltende Rechenstunden. Wien, Gerold, 1918. 112 pp. Kr. 2.00

III. APPLIED MATHEMATICS.

- AHRENS (C.). See LAUENSTEIN (R.).
- ANACKER (K.). Praxis des Flugzeugbaues. Ein Handbuch der Flugtechnik. 3 Bände. Band 1: Das Flugzeug und sein Aufbau. Berlin, 1917. M. 6.00
- ASTRONOMISCHER Kalender für 1918. 37ter Jahrgang. Wien, Gerold, 1917. Kr. 6.20
- BARUCH (A.). Die Grundlagen unserer Zeitrechnung. (Mathematisch-physikalische Bibliothek, Band 29.) Leipzig, Teubner, 1918. M. 1.00
- BRICK (H.). Die Telegraphen- und Fernsprechtechnik in ihrer Entwicklung. (Aus Natur und Geisteswelt, Band 235.) Leipzig, Teubner, 1918. M. 1.50
- ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band 6: Geodäsie und Geophysik, und Astronomie. 1ter Teil, Abteilung B, Heft 4. Leipzig, Teubner, 1918. M. 6.40
- EXNER (F. M.). Dynamische Meteorologie. Leipzig, Teubner, 1917. Gr.8vo. 9 + 308 pp. Geb. M. 16.50
- FINSTERWALDER (S.). Alte und neue Hilfsmittel der Landesvermessung. Festrede. München, 1918. M. 1.00
- HAUBER (W.). Die Grundlehren der Statik starrer Körper. 3ter Neudruck. Leipzig, 1918. M. 1.00
- HEGNAUER (H.). Schulzeichnen auf Grund elementarer Perspektive. Leipzig, Teubner, 1918. 9 pp. + 18 Tafeln. M. 5.00

- HENSELING (R.).** Sternbüchlein für 1917. Mit einem Beitrag von H. H. Kritzinger. Stuttgart, Franckhsche Verlagshandlung, 1917. 8vo. 86 pp. Geh. M. 1.00
- KOPPE (M.).** Die Bahnen der beweglichen Gestirne im Jahre 1918. Eine Tafel nebst Erklärung. Berlin, Springer, 1918. Geh. M. 0.60
- KRITZINGER (H. H.).** See HENSELING (R.).
- LAUENSTEIN (R.).** Die Festigkeitslehre. 13te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1917. M. 5.40
- . Die Mechanik. 10te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1917. M. 5.40
- . Die graphische Statik. 13te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1918. M. 5.40
- LECORNÜ (L.).** La mécanique. Les idées et les faits. Paris, Flammarion, 1918. 16mo. 304 pp. Fr. 4.75
- ROHR (M. v.).** Die optischen Instrumente. (Lupe, Mikroskop, Fernrohr, photographisches Objektiv und ihnen verwandte Instrumente.) 3te, vermehrte und verbesserte Auflage. Leipzig, 1918. 137 pp. M. 1.50
- SILBERSTEIN (L.).** Elements of the electromagnetic theory of light. London, Longmans, 1918. 16mo. 48 pp. 3s. 6d.
- TURRIÈRE (E.).** Sur le calcul des objectifs astronomiques de Fraunhofer. Paris, Service géographique de l'Armée, 1918. 8vo. 123 pp.
- USENER (H.).** Der Kreisel als Richtungsweiser. Seine Entwicklung, Theorie und Eigenschaften. München, 1917. M. 8.00
- WEBER (E.).** Der Weg zur Zeichenkunst. (Aus Natur und Geisteswelt, Band 430.) 2te Auflage. Leipzig, Teubner, 1918. 86 pp. M. 1.50
- WOLFF (H.).** Karte und Kroki. (Mathematisch-physikalische Bibliothek, Band 27.) Leipzig, Teubner, 1917. M. 1.00

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The following dates have been fixed for the meetings of the Society :

Chicago: April 4-5, 1919.

New York: April 26, 1919.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

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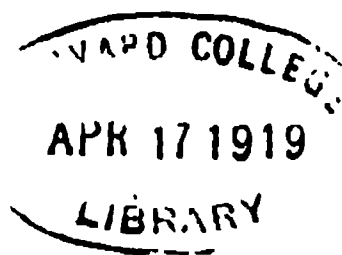
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MATHEMATICS IN WAR PERSPECTIVE.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 27, 1918.

BY PRESIDENT L. E. DICKSON.

AN army officer of high rank, now facing the problems involved in stopping the huge war machine which he had helped to build, recently remarked to me that this getting out of war is far more trouble than getting into it. The armistice has put me in the same boat and, for the purposes of this address, came a few weeks too soon. I had already put myself under obligations to numerous friends, including two in England and France, for furnishing me authoritative information on the rôle of mathematics and its applications in the war. While this information is fortunately no longer needed for its initial purpose, it bears on the timely question of the kind of preparedness which the nation should adopt. While science has played an important rôle in this war, it would undoubtedly play a dominant rôle in a future war, and no scheme of national preparedness will prove adequate which does not insure an ample supply of highly trained scientists and furnish to all men effective training in the fundamentals of exact science. Owing to its recognized value as a fundamental part of military education, I expressly include mathematics, especially trigonometry and graphical analysis. Let it not again become possible that thousands of young men shall be so seriously handicapped in their army and navy work by lack of adequate preparation in these subjects. Nor should so many instructors in courses for prospective officers again be chosen from those who had just passed hastily through the course, not to count those who had merely taken a few private lessons. Fortunately the more widespread and more effective scientific training here advocated as an essential part of national preparedness for war furnishes at the same time the surest means to retain and increase our material prosperity, to add to our health, comfort and conveniences, and so to train our youth in the unravelling of the mysteries of the universe and in habits of drawing accurate conclusions from correctly observed facts that they may the more surely become sane, reliable and efficient citizens.

For data on the military and naval instruction in France during the war, I am indebted to the distinguished mathematician, M. Edmond Maillet, President of the Mathematical Society of France, who also kindly sent me current programmes of requirements for admission to the various schools. As a background we need some facts concerning the instruction given just prior to the war. Cadets to become officers of infantry or cavalry took a two-year course (which was suspended during the period of the war) at the Ecole Spéciale Militaire de Saint-Cyr, the entrance examinations being on algebra (through quadratics), geometry, trigonometry, conic sections, notions of derivatives, descriptive geometry, mechanics, and general physics and chemistry. But the future officer of artillery or engineering was trained at other schools which required more extensive preparation for entrance. After being able to pass the entrance examinations at Saint-Cyr, he entered a special class at a lycée which continued 8 or 9 months prior to October, 1917, but only $5\frac{1}{2}$ months in 1917-18. For example, in the class of Mathématiques Spéciales, preparing particularly for the Ecole Polytechnique, the number of lessons, each of $1\frac{1}{2}$ to 2 hours, were as follows for the 8 (and $5\frac{1}{2}$) months: trigonometry, (college) algebra, and differential and integral calculus, 47 (36); plane and solid analytic geometry, 45 (33); descriptive geometry, 24 (22); mechanics, 20 (7); supplemented by written exercises, quizzes and 12 drawings in descriptive geometry. In such a lycée, he continued also his study of general physics and chemistry, history, and English or German.

The special aim of the famous Ecole Polytechnique is to provide the training in the exact sciences which is necessary for artillery officers and for the various types of engineers in their subsequent technical course at one of the various state schools of applied science, mentioned below. The subjects taught in the two-year course and the number of lectures are as follows:* higher analysis (65), projective, infinitesimal and cinematic geometry (26), application of descriptive geometry to stereotomy (12), mechanics (74), drawing (30), architecture (12), astronomy and geodesy (17), physics (60), chemistry (60), history and literature (40), political and social economy (20), German, English, military science (30), and drill (one hour a week).

* *Journ. de l'Ecole Polyt.*, ser. 2, cah. 14, 1910, pp. I-XLIII.

Among the schools of applied science* are the Ecole des Ponts et Chaussées† for engineers of railroads, ports, rivers, harbors, drainage, etc.; Ecole du Génie Maritime for marine engineers; Ecole d'Hydrographie for a small number of hydrographic engineers, at each of which there is a two-year course primarily for those who have completed the course at the Ecole Polytechnique. The Ecole Nationale Supérieure des Mines trains especially mining engineers, but also for industrial positions; the engineers are taken exclusively from the Ecole Polytechnique and are given a three-year course. The Ecole Centrale des Arts et Manufactures offers a three-year course for engineers for all branches of industry, architecture, mining, machinery, chemistry, etc. The above schools are all at Paris.

The Ecole Navale at Brest offers to candidates of ages 16 to 19, who have had the equivalent of the above special lycée course (with omission of descriptive geometry and electricity in 1918), a two-year course for line officers in the navy. The subjects studied are analysis, rational mechanics, astronomy, navigation, naval architecture and machines, drawing, photography, physics, chemistry, literature, history and seamanship. The course is followed by a cruise of ten months for practical instruction. The entire course has been reduced to five months during the war. To provide further line officers in the navy, the Ecole des Elèves Officiers de Marine at Brest offers a two-year course for enlisted naval men of certain grades and lengths of service who can pass the examinations in arithmetic, (advanced high school) algebra, trigonometry, conic sections, notions of derivatives, elements of plane, solid, and descriptive geometry, mechanics, general physics, geography and French history.

To provide captains and mechanician officers for the merchant marine there are free state schools at 16 French ports, but only those at Havre, Nantes, Marseilles, and Paimpol were open during the war.‡ Beginning with 1918, the course at

* P. Melon, *L'Enseignement supérieur et L'Enseignement technique en France*, ed. 2, 1893.

† Cf. transl. of French report by Major W. D. Connor, *National School of Bridges and Highways*, Paris, France. U. S. Printing Office, 1913, 42 pp.

‡ *Organisation et Fonctionnement des Ecoles d'Hydrographie et de l'Institut Maritime du Havre; conditions d'admission, programmes des examens pour . . . capitaine au long cours*, Paris, A. Challamel, 1918, 108 pp. *Programme . . . d'officier mécanicien de la marine marchande*, Paris, Vuibert, 68 pp.

the first three of these four schools will require two years. For the special "brevet supérieur" captain or mechanic, the examinations include also differential and integral calculus, rational mechanics, physics, elementary chemistry, machines, etc. There are schools for apprentice mechanics at Lorient, Brest, Toulon, and Havre.

To supply the increased need for army officers during the war, the Ecoles Militaires d'Aspirants at Saint-Maixent, Saumur, Fontainebleau and Versailles for the infantry, cavalry, artillery and engineers, respectively, provided practical courses of five months for sub-officers, regarded as capable of becoming officers, who passed oral and written examinations in arithmetic, plane geometry, linear equations in several unknowns, definitions of the trigonometric functions and of the terms in solid geometry, formulas for surfaces and volumes (without proofs), elementary notions in physics and chemistry, geography, French history and literature. Also an extensive acquaintance with descriptive geometry was required of candidates for entrance to the school for engineers. No examination was required in the case of sub-officers who had served 15 months in the army and were recommended by the military authorities, nor of those who had been admitted to the Ecole Polytechnique or the school at Saint-Cyr.

In England the training of naval cadets under the system* adopted in 1913 consisted of a two-year course at the Royal Naval College at Osborne, followed by a two-year course at the Royal Naval College at Dartmouth, and six months on a training cruiser. In 1912 there were 439 cadets at Osborne and 406 at Dartmouth. To enter Osborne the candidate must be between $12\frac{2}{3}$ and 13 years of age and pass an entrance examination in arithmetic, algebra (linear equations in one or more unknowns), geometrical constructions and the substance of the first book of Euclid, history, geography, and languages. In arithmetic, algebra, geometry and the elements of plane trigonometry, there are $6\frac{3}{4}$ hours per week of instruction and 2 of preparation during the first four terms, each of twelve weeks, and $7\frac{1}{2} + 1\frac{1}{4}$ during the last two terms. At Dartmouth all cadets take algebra, plane and solid geometry and plane and spherical trigonometry, while the more proficient men take also analytic geometry and elementary notions of

* Great Britain Admiralty Committee on education and training of naval officers, Accounts and Papers, Navy, vol. 43, 1913, 170 pp.

calculus. To insure that all shall attain to the standard in navigation, extra time is provided in navigation for the weaker cadets at the expense of their further progress in mathematics. In the first two terms there are $5 + 2\frac{1}{2}$ hours of mathematics and no navigation, while the later schedules are

	3d term	4th term	5th term	6th term
Math.	4+2	4+2½ or 3 +2	4+2½ or 3+2	4+2½ or 3+2 or 2+1½
Nav.	1+½	2+1 or 1½+½	2+1 or 3+1½	2+1 or 3+½ or 4+2

During the subsequent 24 weeks on a training cruiser, five hours were devoted daily to study; the time for the optional course in trigonometry and calculus was included in the six hours per week assigned to navigation.

Professor W. Burnside has kindly provided me with a statement of the instruction during the war. The only training in navigation for prospective officers of the British navy is given on board ship and at the Royal Naval College, Dartmouth. There the course was reduced to five terms by cutting down somewhat the non-professional subjects such as history. During the initial two terms, two hours per week were given to recitation and $\frac{1}{2}$ hour to preparation in navigation. In the third term the time ranged from $2 + 1$ hours for the best to $3 + 1\frac{1}{2}$ for the poorest of the six classes into which the cadets were separated on the basis of ability. In the fourth term the hours were $2 + 1\frac{1}{2}$ to $3 + 2$; in the final fifth term, $3 + 1\frac{1}{2}$ to $4 + 2$. The only text-book used is S. F. Card's Navigation Notes and Examples, 245 pages, second edition, 1917, Arnold (to be had of Longmans, Green and Company in America), but reference was made to Chapter XVII of the Admiralty Manual of Navigation, 525 pages, 1914, London (3 shillings). Cadets procured Inman's tables and the Nautical Almanac for 1919, and had access to Burdwood's Azimuth Tables. They acquired facility in using parallel rulers, dividers, sextants, magnetic and gyro compasses, station pointers, and the large scale chart S 389 D.

For artillery officers in the army and for gunnery officers in the navy, the whole of the theoretical and a great part of the practical training is given at the Ordnance College at Woolwich, where most of the courses are technical rather than theoretical. The only course which gives teaching on theoretical ballistics, external and internal, is one of 9 hours a week for

8 months for artillery officers who are practical experts of at least six years' experience in their profession. The books used are the Army and the Admiralty Gunnery Manuals which are strictly confidential documents. There are no public British text-books on ballistics.

The work at our own government schools* at West Point and Annapolis need not be reported on here. The emphasis during the first two years is on pure mathematics. There are various post-graduate army schools. In marine engineering the first year's post-graduate work is done at the Naval Academy and the second year at Columbia University. Other graduates, who are to become naval constructors, take a three-year course at the Massachusetts Institute of Technology. On account of the war, a class recently graduated at West Point after two years' work. Annapolis is temporarily on a three-year schedule, the enrollment in the entering, middle and graduating classes being now 963, 678 and 485 respectively, with 18 men per section in mathematics.

The Naval Auxiliary Reserve Training Schools at Pelham Bay and at the Municipal Pier of Chicago taught navigation, seamanship, etc., to a large number of enlisted men seeking an ensign's commission. As many men lacked adequate mathematical preparation for the work at the latter school, preliminary courses in trigonometry and navigation were given to about 900 of these men at the University of Chicago and to a like number at Northwestern University.

At the Officer Material School, Cambridge, Mass., the Navy conducted, during the past sixteen months, four-month courses in navigation (including plane and spherical trigonometry), ordnance and gunnery, seamanship, and naval regulations, the number of hours per week of class work being 8, 8, 7, and 2, respectively. The 1,000 students were selected from those enlisted in the navy and were all above 21 years of age. The instructors expressed belief in the need in the future of more thorough grounding in mathematics, up to and including trigonometry.

The U. S. Shipping Board conducted free schools for the

* Besides their annual registers, see the report of the International commission on the teaching of mathematics, U. S. Bureau of Education, Bulletin, 1912, No. 2; also, Report of U. S. Commissioner of Education, 1913, I, pp. 599-630. In the *Amer. Math. Monthly*, Oct., 1918, pp. 370-2, Professor Root described the temporary course of 16 weeks in navigation at Annapolis for reserve officers.

training of navigation and engineer officers for the merchant marine. The first schools were opened July 1, 1917, and others from time to time up to November, 1918,—31 schools in all. The total attendance has been 12,218; of these, 3,509 completed the course for deck officers and 3,290 the course for engineer officers. For the navigation schools the prerequisite was two years' sea experience and graduation from a grammar school. The instruction (30 hours per week in class and 10 of other study) was on navigation by dead reckoning and by observation (Bowditch) and in "Rules of the Road" (published by the Hydrographic Office), international code of signals, etc. Each school possessed six sextants, a chronometer, compass, azimuth instrument, and five dividers. In the schools for engineer officers the course of one month (36 hours a week) covered the technical knowledge required for the grade of Chief. The text-books were Dyson's Practical Marine Engineering, and the Crosby Company's Practical Instructions on the Steam Engine Indicator. More difficulty was found with mathematics than anything else and special instruction was given in mathematics in the early part of the course. On the average there was one instructor for ten students in these schools. For the preceding information I am indebted to Director Henry Howard and his assistants.

The Student Army Training Corps was formally inaugurated on October 1, 1918, at some 500 colleges and universities. By November 1 the enrollment had reached the following figures obtained from the War Department: army, collegiate section, 127,766; vocational section, 37,261; navy, 12,598; marine, 413; in process of induction, 976. The following conclusions are based upon 29 replies to a questionnaire from 17 large universities and 12 colleges and small universities, which together represent all parts of the country. At these schools, 14,785 men took trigonometry in 522 sections with an average of 28 men and four hours of class recitation. As 39 per cent. of the army and naval enrollment (exclusive of the vocational section) at these schools took trigonometry, probably about 55,000 of the S. A. T. C. men in all the colleges took that subject. Excluding the classes in trigonometry conducted for naval men only, there were 257 instructors, 63 per cent. of whom were on the regular mathematical staffs, 20 per cent. were from other departments, and 17 per cent. were temporary appointments. The replies indicated unanimously that

the work in trigonometry was not as efficient as in their usual classes, nor in 75 per cent. of the institutions as extensive as usual. At all but two of the 29 schools, the chief reason assigned for the inferior work was absence for military duty, while poorer preparation was assigned as one of the reasons at half the schools and the influenza at several. At 18 of these 29 schools, surveying was taught to 3,664 S. A. T. C. men on an average of six hours per week of field work and three of indoor class work; the work was neither as effective nor as extensive as usual, due only partly to poorer preparation, but unanimously and emphatically attributed to cuts for military duty. On the average there were four men in a squad and 26 men per instructor; only 40 per cent. of the instructors were on the regular staffs. Navigation was taught to 2,170 men in sections of usually not over 25 men and with four recitations a week. Smaller numbers took courses in firing data, gunnery, ballistics, aerodynamics, statistics, as well as various subjects in mathematics. But the scheme partially failed because the lack of available experienced officers required that its execution be left usually to officers of very recent vintage, who were unable to understand why other young prospective officers needed the college courses, even though prescribed by the War Department, and, instead of regarding cuts from classes as a breach of military discipline, proceeded to remove men from their classes and assign them to all sorts of minor, menial, and clerical duties.

The ultimate object of exterior ballistics is to obtain data for range tables and the various ballistic corrections for practical use in directing the fire of the gun. When we entered the war we had no range tables for various types of guns we decided to adopt, especially for the anti-aircraft guns. The construction of the necessary new range tables involved not only the obtaining of a vast amount of experimental data, but also the elaboration of the theory of the differential equations which takes into account not only the resistance of the air but also its temperature and its decrease in density at higher altitudes, as well as corrections for the wind. Under the leadership of two of our well known mathematicians, Professors F. R. Moulton, and O. Veblen, now Majors in the Ordnance Department, two groups of mathematicians including Alexander, Bennett, Blichfeldt, Bliss, Buck, Dines, Gronwall, Hart, Haskins, Jackson, MacMillan, Milne, H. H. Mitchell, Ritt, Roever,

H. L. Smith, and Vandiver, have been engaged in this important work at Washington and Aberdeen, Md. For given initial conditions as to the gun, ammunition, elevation, and on the assumption of normal air density and no wind, the trajectory is now computed in about half a day, with a gain in accuracy. Some further simplification appears to result from the use of the adjoint system of differential equations. But it would be foolish for me to attempt to go into details since you are to have the pleasure of hearing this afternoon five ballistic experts who come to us direct from the two centers of ballistic work in America. Following my request for some information suitable for use in this address, I received from both Washington and Aberdeen huge bundles of blue prints showing hundreds of beautiful trajectories and other curves and many heavy mathematical manuscripts,—a sort of long range bombardment which it seemed the part of wisdom to dodge and trust to the direct fire of the newly arrived experts.

Believing that navigation should receive more attention in future in collegiate instruction, I shall give an outline of certain mathematical aspects of the subject.

By dead reckoning is meant the determination of the position of a ship by means of the measured distances and courses which it has sailed from a known position P . The true course C is the angle made by the ship's track with the north and south line. The method of plane sailing is employed when the distance D sailed is so short that we may neglect the curvature of the earth. Hence we have a plane right triangle with hypotenuse D , one angle C , the vertical leg being the difference of latitude expressed in nautical miles, and the horizontal leg being called the departure, as in surveying. Thus the legs are

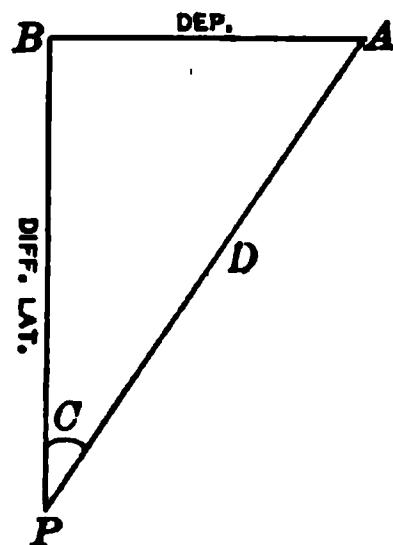


FIG. 1.

$$\text{Diff. Lat.} = D \cos C, \quad \text{Dep.} = D \sin C,$$

and may be computed by logarithms or found as in surveying by inspecting a traverse table in which are entered the products of each number $D = 1, 2, \dots, 600$ by $\cos C$ and by $\sin C$ for $C = 1^\circ, 2^\circ, \dots, 89^\circ$. It remains to find the difference of longitude, i. e., the arc of the equator intercepted by the me-

ridians through P and A , the point from which we sailed and the point arrived at. The east and west arc through A which is intercepted by those two meridians is the departure. Since these two arcs subtend equal angles at their centers, their ratio equals the ratio of the radii of their circles, which is immediately seen to be the secant of the latitude. Hence $\text{Dep.} = (\text{Diff. Long.})(\cos \text{Lat. } A)$, from which we may find the difference of longitude by logarithms or by a traverse table.

In middle latitude sailing we take into account the curvature of the earth and assume that the ship's track is a rhumb or loxodromic line making the same angle C with all the meridians crossed. Divide the distance D into parts each so small that it can be regarded as the hypotenuse of a plane right triangle with an angle C . The sum of the vertical legs of these small triangles is seen to equal the difference of latitude, so that we again have $\text{Diff. Lat.} = D \cos C$. Since the meridians converge towards the north pole, the sum of the east and west legs of our small triangles has a value which exceeds the departure at A and is less than the departure at P and is assumed to equal the departure in middle (or mean) latitude, i. e., the east and west arc intercepted by the meridians through P and A on the parallel of latitude half way between the parallels of latitude at P and A . If these parallels are far apart or if either is near a pole, the assumption just made introduces too large an error. When the assumption is valid, we have $\text{Dep. in Middle Lat.} = D \sin C$. Hence we may proceed exactly as in plane sailing with departure replaced by departure in middle latitude.

Mercator's sailing involves no assumption restricting its accuracy and has the further advantage that the computations can be conveniently checked graphically on a chart which shows the ship's position at all times and hence its relation to possible danger points. The earth's surface is mapped on the interior of a rectangle in such a way that the meridians are represented by parallel straight lines, as also are the parallels of latitude. Since the rhumb line on which we sail crosses all the meridians at the same angle C , it is mapped as a straight line. The act of plotting as straight lines the earth's meridians which converge at the poles has caused an opening out of these meridians, i. e., a stretching of east and west lengths. But we desire that any small figure on the map shall be of the same shape as the corresponding figure on the earth, even

though it be magnified. Hence there must be simultaneously a stretching of north and south lengths, the amount of stretching being the secant of the latitude. If L is the latitude of a point on the earth, the number m of nautical miles in the magnified latitude (called meridional parts) is given by a table, which is computed by use of the formula

$$m = r \log \tan (45^\circ + \frac{1}{2}L) \\ - r(e^2 \sin L + \frac{1}{3} e^4 \sin^3 L + \frac{1}{5} e^6 \sin^5 L + \dots),$$

where r is the equatorial radius and e is the eccentricity of the ellipse whose rotation produces the earth's surface, while Napierian logarithms are employed. Taking for simplicity the earth to be a sphere, a small length $r dL$ on a meridian is represented on Mercator's map by $r \sec L dL$, whence the length on the map of the meridian from the equator to latitude L is

$$\int_0^L r \sec L dL = r \log \tan (45^\circ + \frac{1}{2}L).$$

By making use of the table of meridional parts we can readily construct to scale a rectangular Mercator's chart showing for example the parallels of latitude for $20^\circ, 21^\circ, \dots, 30^\circ$ North latitudes and the meridians for $70^\circ, 71^\circ, \dots, 85^\circ$ West longitudes; the entire rectangle is therefore divided into 10×15 small rectangles with equal bases, but varying heights which increase as we pass to higher latitudes. Such position charts are published by the U. S. Hydrographic Office. On a Mercator map, angles are the same as the represented angles on the earth, and difference of longitude is found by the scale at the bottom of the large rectangle. But as distances and differences of latitude appear magnified, the lines representing them are measured to the scale appropriate to their latitude, such varying scales being often given in the right and left hand margins directly opposite to the latitude.

For the computation by logarithms or a traverse table, we use the plane right triangle on a Mercator's map whose legs are the meridional difference of latitude and the difference of longitude and one angle is the course C , as well as the formula $\text{Diff. Lat.} = \text{Dist.} \times \cos C$ derived above from the curvilinear triangle on the earth. We make no use of the side "departure" in the last triangle, or of the hypotenuse of the former.

Great circle sailing is employed on very long voyages since the distance sailed is then a minimum, although it has the inconvenience that the course is changing continually. We first plot the track from port to port on a gnomonic projection chart (a projection on a tangent plane from the center of the earth), on which the great circle track is represented as a straight line. Then we transfer the route to a Mercator's map. In 1858, Sir George Airy proposed a method of representing approximately a great circle directly on a Mercator's map (Bowditch, page 82). Great circle distances and courses are found by spherical trigonometry, as in the later discussion of the astronomical triangle. An account of the literature on great circle sailing has been given by G. W. Littlehales.*

Owing to various inaccuracies in the data used in dead reckoning, the navigator must correct his estimated position by use of sights or observations of the sun or stars. We proceed to explain the method now in general use.

Suppose that a navigator measures with a sextant the sun's altitude (its angle of elevation above the horizon) and finds it to be 70° , so that the sun's zenith distance z is 20° . Then he is 20° or 1,200 nautical miles from the geographical position U of the sun, i. e., the point on the earth having the sun in its zenith. Hence the ship lies on a small circle whose spherical radius is 1,200 miles and spherical center is U . This circle of equal altitudes is in practice replaced by the tangent line, called a Sumner line of position, which is perpendicular to the bearing of the sun. It was discovered in 1837 by an American shipmaster, Capt. T. H. Sumner,† under the stress of saving his ship from imminent danger. Two special cases of the method had long been in constant use. The navigator took a sun sight just after sunrise and just before sunset to determine his longitude, the Sumner line then being perpendicular to the approximately East or West bearing of the sun. He took a noon sight to find his latitude, the Sumner line then being perpendicular to the North or South bearing of the sun.

In 1875 Admiral Marcq Saint Hilaire‡ of the French navy gave the following method to find the Sumner line. Given

* The Development of Great Circle Sailing, U. S. Hydrographic Office, 1889, No. 90.

† A New and Accurate Method of Finding a Ship's Position at Sea, Boston, 1843; third ed., 1851.

‡ Calcul du point observé, *Revue Maritime et Coloniale*, vol. 46, 1875, p. 341, p. 714.

the estimated position A of the ship as found by dead reckoning and the geographical position U of the sun (or a star), we compute the great circle distance AU by one of the formulas below, and either compute the bearing (azimuth) of U from A or take it from a table of azimuths or from Weir's Azimuth Diagram. Then $h = 90^\circ - AU$ is the computed altitude. Let h' be the sun's altitude observed with the sextant. On a Mercator's chart lay off from A the difference of the altitudes in miles towards U or in the opposite direction from U according as h is less than or greater than h' . Then the straight line through the point B thus determined and at right angles to the bearing is the Sumner line* containing the ship's true position.

The computation is made by use of formulas derived from the astronomical triangle MPZ , whose projection on the plane of the horizon is shown in Fig. 2, in which M represents the sun (or star), P the elevated pole, and Z the observer's zenith (point overhead). The declination d of the sun at the moment

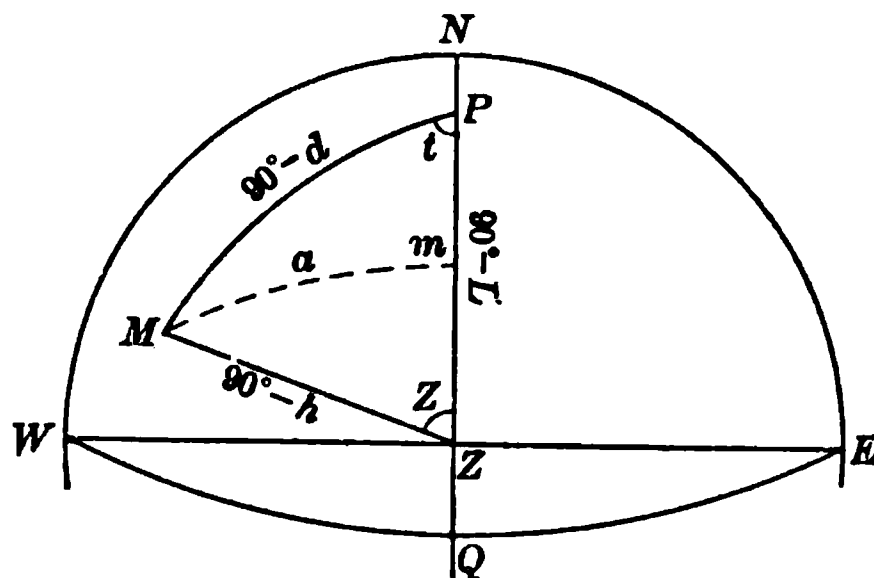


FIG. 2.

of observation is given by the Nautical Almanac; its hour angle t and the observer's latitude L are supposed known. A standard formula of spherical trigonometry expresses $\cos MZ$ in terms of the remaining two sides and their included angle t :

$$\begin{aligned} \cos (90^\circ - h) &= \cos (90^\circ - L) \cos (90^\circ - d) \\ &\quad + \sin (90^\circ - L) \sin (90^\circ - d) \cos t. \end{aligned}$$

Replacing $\cos t$ by $1 - 2 \sin^2 \frac{1}{2}t$, we get

$$\sin h = \cos (L - d) - 2 \cos L \cos d \sin^2 \frac{1}{2}t.$$

* Approximately. See the report below on Guyou's tables.

By finding the final product by logarithms, we readily get h . It is customary to use a formula obtained from the last by introducing versine x for $1 - \cos x$ and haversine x for $\frac{1}{2}$ vers $x = \sin^2 \frac{1}{2} x$. Since $h = 90^\circ - z$, we get

$$1 - \text{vers } z = 1 - \text{vers } (L - d) - 2 \cos L \cos d \text{ hav } t,$$

whence, by cancellation and division by 2, we finally have*

$$\text{hav } z = \text{hav } (L - d) + \cos L \cos d \text{ hav } t.$$

Table 45 in Bowditch's American Practical Navigator gives the haversines and their logarithms of angles at intervals of 15 seconds of angle (or one second of time) up to 120° (or 8 hours), with an extension to 180° . Since we now have the three sides of our triangle MPZ , we may compute the azimuth angle Z by use† of

$$\cos^2 \frac{1}{2} Z = \cos s \cos (s - p) \sec L \sec h, \quad s = \frac{1}{2}(L + h + p),$$

where p is the polar distance $90^\circ \pm d$.

In 1875, Lord Kelvin‡ stated that it ought to be the rule and not the exception to use Sumner's method for ordinary navigation at sea.

We may solve our astronomical triangle PZM by use of the spherical traverse table published by Commander F. Radler de Aquino§ of the Brazilian Navy. In Fig. 2, let the perpendicular a from M to PZ divide the latter into the parts $Pm = 90^\circ - b$ and $Zm = 90^\circ - B$. Use is made of six formulas given by Napier's rules. By means of

$$\sin d = \cos a \sin b, \quad \cot t = \cot a \cos b,$$

* C. L. Poor, in his *Simplified Navigation for Ships and Aircraft*, 125 pp., 1918, N. Y., The Century Co., describes his machine to compute z by this formula. It is in effect a circular slide rule with several circular discs for finding the separate terms of the formula.

† Or by a formula for its haversine, Muir, *Navigation*, 1918, p. 444; Card, *Navigation Notes*, p. 9 (example, p. 90).

‡ Popular lectures and addresses by Sir Wm. Thomson, vol. 3, "Navigational Affairs," 1891 and 1894, Macmillan and Co.

§ The "Newest" Navigation. Altitude and Azimuth Tables, 2d ed., London, J. D. Potter, 1917. Aquino, *O methodo de Marcq Saint Hilaire*, Rio de Janeiro, 1902; *Typos de calculo* . . . , 1902. The table was reprinted and illustrated by examples (but without explanation of its construction) in *Altitude, Azimuth and Line of Position*, U. S. Hydrographic Office, 1917, No. 200.

he computed and tabulated the values of d and t corresponding to values of b for every degree and of a for every $30'$ from 0° to 84° and for every 1° from 84° to 90° . Since

$$\sin h = \cos a \sin B, \quad \cot Z = \cot a \cos B$$

are of the same form as the preceding equations, the values of h and Z for given a and B are already known. Hence the table has a double set of labels

$$\underline{B \backslash b \mid h \backslash d \mid Z \backslash t \mid}$$

at the top of the page for any given value of a . Finally,

$$\sin a = \cos d \sin t, \quad \cot b = \cot d \cos t$$

show that we have automatically tabulated the values of a and b (marked by labels at the bottom of the page) which correspond to given values of d and t . The table has a column with the heading $C \backslash c$ showing $c = 90^\circ - b$ or $C = 90^\circ - B$; also a column showing the two angles at M . To do away with certain interpolations, we take an assumed latitude and longitude nearly the same as those given by dead reckoning, without changing the accuracy of the Sumner line.

Lord Kelvin* had previously published a smaller table of the same type.

F. Souillagouët's† final table of 108 pages is a traverse table for the right spherical triangle MmP of Fig. 2. It gives as entries $\phi = Pm$ and a for arguments t (angle at P) and hypotenuse $90^\circ - d$, each at intervals of $30'$ up to 90° . Having a and $mZ = PZ - \phi$, we again enter the table and read off the azimuth Z and altitude h . His first table of 254 pages serves to solve triangle PZM ; let $\phi' = ZK$ be the perpendicular from Z to PM , and let $\phi = PK$. Then

$$\begin{aligned} \tan \phi &= \cot L \cos P, & \sin h &= f \cos(90^\circ - d - \phi), \\ f &= \sin L / (\cos \phi) = \cos \phi'. \end{aligned}$$

The table gives ϕ and $\log f$. We then get h .

* W. Thomson, *Tables for facilitating of Sumner's method at sea*, London, 1876. Cf. H. Jacoby's *Navigation*, 1917, pp. 126–133, 292–317.

† *Tables du point auxiliaire pour trouver rapidement la hauteur et l'azimut estimés, suivies d'un recueil nouveau de tables nautiques . . .*, new ed., Toulouse, 1900.

G. W. Littlehales* published a book of charts which serve to solve graphically the astronomical triangle PZM . Employ a stereographic projection of the celestial sphere on the plane of the observer's meridian (a projection from the pole of the meridian circle). By use of the latitude, $90^\circ - PZ$, mark the observer's position Z on the bounding meridian. Locate the position M of the observed celestial body by means of its declination $90^\circ - PM$ and its hour angle MPZ . In the triangle

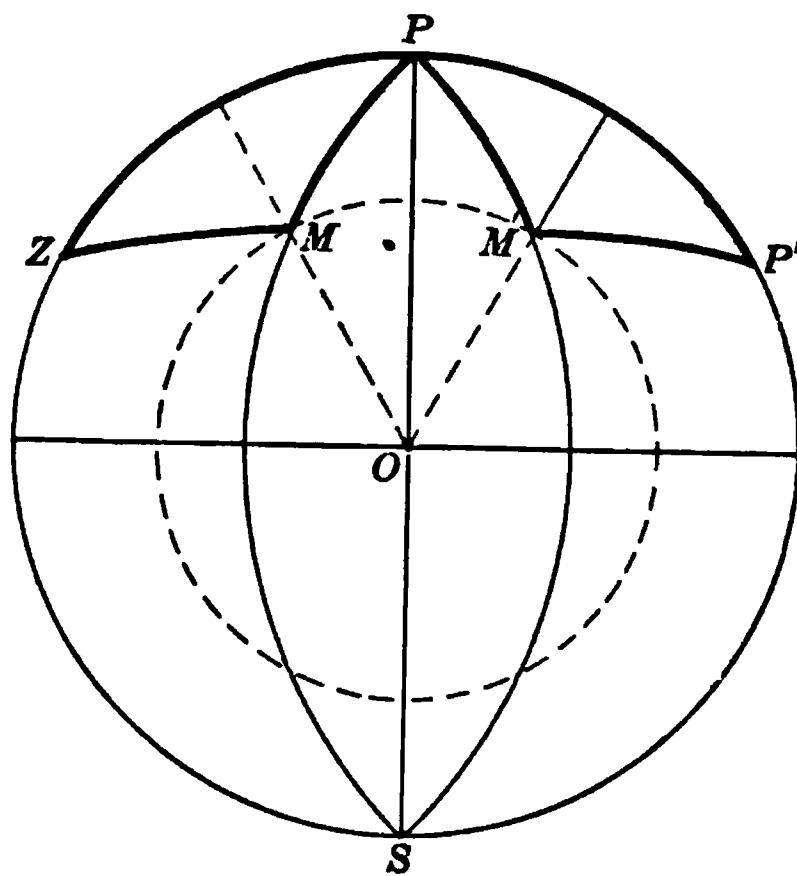


FIG. 3.

PMZ we have two sides and the included angle and desire the azimuth *PZM* and co-altitude *ZM*. Rotate the triangle about the center *O* of the projection with the side *PZ* kept in coincidence with the bounding meridian until *Z* is brought to the position of *P*, whence *P* is moved to a position *P'*, and *M* to *M'*. The co-altitude now lies along a meridian *PM'* and the azimuth becomes the angle *M'PP'* between two meridians, so that each can be measured by means of the graduations of the projection. To obviate the necessity for the actual rotation of the triangle, draw a series of equally spaced circles with the center *O*, numbered serially from *O* outward, and a series of equally spaced radial lines, marked by numbers indicating their angular distances in minutes of arc counted in clockwise direction from *OS*. After plotting *M*, note the num-

* Altitude, Azimuth and Geographical Position, comprising Graphical Tables . . . , Philadelphia, Lippincott, 1906, \$25.

ber of the circle and the number of the radial which pass through M . To the number of the radial add the number of minutes in the co-latitude. The sum is the radial number of M' , which is thus located in the circle just noted. We can now read off from the graduated arcs of the projection the desired altitude and azimuth. The projection was constructed for a sphere 12 feet in diameter and subdivided into 368 sections. The plate for a section is about a foot square and two of them are printed on the large page. There is a diagram which furnishes an index to the plates. For example, let the declination be $45^{\circ} 54' N$, the hour angle $30^{\circ} 43.5'$, and the latitude $39^{\circ} 16' N$. Plotting the declination and hour angle roughly on the index of plates with reference to the parallels and meridians (counted from the left-hand bounding meridian), we find that the position of the observed body falls on plate No. 63 approximately at the intersection of circle 17.2 with radial 8,400. The co-latitude is 3,044 minutes. Thus the approximate position of the rotated position M' is the intersection of circle 17.2 with radial $8,400 + 3,044$ and hence falls on plate No. 258. Turn to plate No. 63 and plot the declination and hour angle carefully; we find that M is at the intersection of circle 495.6 with radial 8,411. Then M' on plate No. 258 is at the intersection of that circle with radial $8,411 + 3,044$, whence we read off the altitude $66^{\circ} 36'$ and the azimuth $N 63^{\circ} 32' W$. The method applies at once to sailing on a great circle from Z to M , the initial course being angle PZM . In the problem to identify an observed star, we know its altitude and azimuth and hence point M' ; we get M and hence its declination and hour angle.

E. Guyou* recently published extensive tables for the accurate simultaneous determination of altitude and azimuth. Underlying his method are geometrical facts of considerable interest. On the sphere let CC_1M be a circle of position with the center A and let Z be the position of the ship estimated by dead reckoning (Fig. 4). Let I be the intersection of the circle with the great circle AZ . On Mercator's chart, let cc_1m and zia be the curves which represent the circle CC_1 and the great circle ZIA (Fig. 5). The true line of position ih is the normal at i to the arc zi . But by Saint Hilaire's method we draw the tangent zj at z to the arc zi and take a perpendic-

* Nouvelles Tables de Navigation, Paris, 1911. Vol. 1, Réduction à l'Equateur, 33 + 370 pp. Vol. 2, Calcul de la Hauteur et de l'Azimut, 284 pp.

ular jh' to this tangent as the line of position. While this line passes very near to i , its direction is in error by an angle hjh' equal to the angle between the tangents at z and i to the

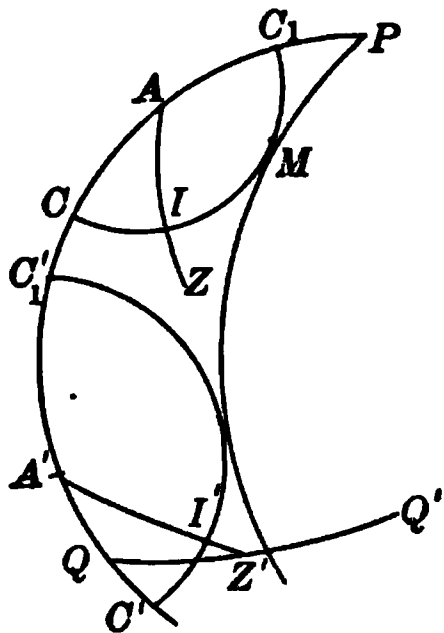


FIG. 4.

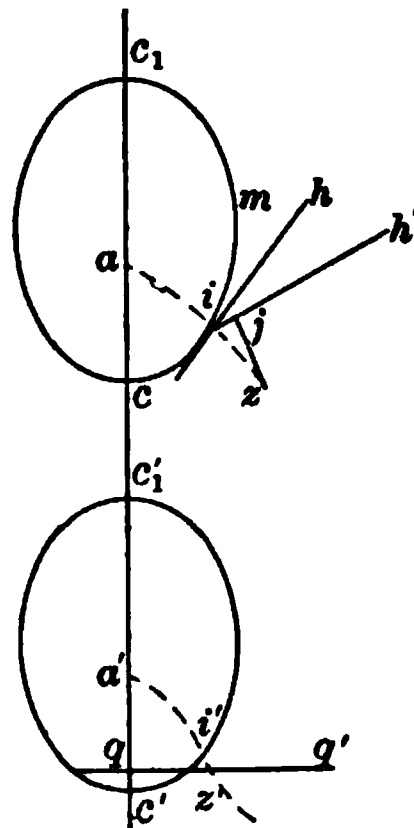


FIG. 5.

arc zi . This error increases with the latitude and practically disappears at the equator, i. e., in the case of circle $C'C_1'$ and point Z' of Fig. 4, since the great circle arc $Z'I'A'$ is represented on the chart (Fig. 5) by a curve $z'i'a'$ which has an inflexion at z' and hence coincides with its tangent for a considerable length. The last fact is the basis of Guyou's method to find a line of position which presents all the advantages of the line of Saint Hilaire and yet is free from the imperfections with which the latter line is in general affected. Starting with the "figure" (cc_1, z) composed of the curve cc_1 and the point z , slide it down to occupy the position $(c'c_1', z')$. This displaced figure represents on the sphere a figure composed of a circle $C'C_1'$ and point Z' , for which Saint Hilaire's method is practically exact, since $z'i'a'$ is practically straight near z' . The method consists of two operations,—reduction to the equator and determination of the altitude and azimuth for the reduced figure at the equator, being accomplished by tables 1 and 2 respectively. First, let

$$H = 90^\circ - CA, D = 90^\circ - PA, P_e = ZPA, L = 90^\circ - PZ$$

be the true altitude, declination and hour angle of the observed

body and latitude of the estimated position Z , and hence the "real data." Let the corresponding values for figure $(C'C_1', Z')$, with Z' on the equator, be

$$H' = 90^\circ - A'C', \quad D' = QA', \quad P_e' = P_e = Z'PA', \quad L' = 0,$$

which are the "reduced data." By use of relations like

$$qc = \log \tan \left(\frac{\pi}{4} + \frac{QC}{2} \right),$$

it is proved in his article* (but not stated in his book) that

$$\cot \frac{H' + D'}{2} = t \cot \frac{H + D}{2}, \quad \cot \frac{H' - D'}{2} = \frac{1}{t} \cot \frac{H - D}{2},$$

$$t = \tan \left(\frac{\pi}{4} + \frac{L}{2} \right).$$

Table I gives the resulting values of $\frac{1}{2}(H' \pm D')$ as functions of $H \pm D$ for each L . By use of a right triangle, we get $\tan Z_e = \cot D' \cdot \sin P_e$, where Z_e is the azimuth. Table II gives the values of H and Z as functions of D' .

SOME FURTHER BOOKS ON NAVIGATION.

If all the books on navigation were collected together they would sink a ship. The following books in English are not afraid† of a needed mathematical formula and appear to be

* "Nouvelle méthode pour déterminer les droites de hauteur et le point observé," *Revue maritime*, vol. 180, Feb., 1909, pp. 223-266. Also, *Rivista marittima*, Aug., 1909.

† The following avoid explicit formulas: H. Marshall, *Nav. made easy*, Milwaukee, 1877. W. C. Bergen, *Practice of Nav. and Naut. Astr.*, 671 pp., 9th ed., 1902, North Shields. L. Young, *Simple Elements of Nav.*, 248 pp., 2d ed., 1898, N. Y. A. C. Johnson, *On Finding the Lat. and Long. in Cloudy Weather*, 39 pp., 22d ed., 1900 (and 1903), London, Potter; and a *Handbook for Star Double-Altitudes*, 32 pp., 1898. H. Taylor, *Modern Nav.*, 347 pp., 1904, San Francisco. C. E. MacArthur, *On Nav. simplified*, 121 pp., 1906, Rudder Pub. Co. C. L. Poor, *Naut. Science in its Relation to Practical Nav.*, 329 pp., 1910, Putnam's Sons, \$2. J. Pendlebury, *Plain Everyday Nav.*, 44 pp., 1911, Yonkers, \$2.50. W. J. Smith, *The Self-instructor in Nav.*, 138 pp., 3d ed., 1912, Bellingham, Wash., \$3. H. L. Thompson, *Self-instruction in Nav.*, 80 pp., 1916, Portland, Me., \$1.50. H. Jacoby, *Nav.*, 350 pp., 2d ed., 1917, Macmillan, \$2.25. G. L. Hosmer, *Nav.*, 214 pp., 1918, Wiley, \$1.25. S. T. S. Lecky, *Wrinkles in Practical Nav.*, 18th ed., 1918, Van Nostrand, \$10. F. S. Hastings, *Modern Nav. by Sumner-St. Hilaire Methods*, 84 pp., 1918, Appleton and Co. W. J. Henderson, *Elements of Nav.*, new ed., 1918, Harper and Bros., \$1.25. A. G. Mayor, *Nav. Illustrated by Diagrams*, 207 pp., 1918, Lippincott, \$1.50. The last six books are good of their kind.

suitable as college texts: W. C. P. Muir, *A Treatise on Navigation and Nautical Astronomy*, 784 pages, 4th edition, 1918, U. S. Naval Institute, Annapolis, the text-book at the U. S. Naval Academy (contains 160 pages on the compass and the theory of its deviations). W. R. Martin, *Navigation and Nautical Astronomy*, 429 pages, 3d edition, 1899, Longmans, Green and Company. J. H. C. Coffin, *Navigation and Nautical Astronomy*, 221 pages, 7th edition, 1898, Van Nostrand. F. C. Stebbing, *Navigation and Nautical Astronomy*, 328 pages, 1896 and 1903, Macmillan and Company. J. Gill, *Text-book on Navigation and Nautical Astronomy*, 380 pages, 1898 and 1905, Longmans, Green and Company. W. Hall, *Modern Navigation*, 378 pages, 1904, Clive, London. V. J. English, *Navigation for Yachtmen*, 342 pages, 1896, H. Cox, London. Card's book was cited above.

The following three books are good ones of the Bowditch type: H. Raper, *Practice of Navigation and Nautical Astronomy* (1840), 934 pages, 19th edition, 1899, London, J. D. Potter; M. F. Maury, *A new Theoretical and Practical Treatise on Navigation*, 520 pages, 3d edition, 1864, Philadelphia; J. H. Colvin, *Nautical Astronomy*, 127 pages, 1901, London, Spon.

Among very clear, elementary books are those by W. T. (Earl of) Dunraven, *Self-instruction in the Practice and Theory of Navigation*, Macmillan, two volumes, 1900, three volumes, 1908, \$8; D. Wilson-Barker and W. Allingham, *Navigation, Practical and Theoretical*, 154 pages, 1896, \$1.50, London, Griffin and Company; J. Merrifield, *Treatise on Navigation for Use of Students*, 305 pages, 5th impression, 1900, Longmans, Green and Company (dead reckoning only); F. S. Hastings, *Navigation*, 109 pages, 1917, 75 cents, Appleton (no Sumner line); J. R. Walker, *An Explanation of the Method of Obtaining the Position at Sea known as the New Navigation*, 67 pages, 1901, Portsmouth (Sumner line only). Many problems and examination papers occur in A. P. W. Williamson's *Text-book of Navigation and Nautical Astronomy*, 418 pages, 2d edition, 1915, Portsmouth; and in H. B. Goodwin's *Problems in Navigation*, two volumes, 1893-6, London, Philip and Son. Wentworth's text omits the subject of Sumner lines.

J. B. Harbord's *Glossary of Navigation*, 512 pages, 3d edition, 1897, Portsmouth, is an excellent nautical dictionary and reference book.

Of French books, the two by J. B. Guilhaumon, *Astronomie et Navigation suivies de la Compensation des Compas*, 611 pages, 5th edition, 1912, and *Eléments de Navigation et de Calcul nautique*, two volumes, 1891, and G. Lecoq, *La Navigation astronomique et la Navigation estimée*, 392 pages, 1897, all published by Berger-Levrault, Paris, seem excellent. Especially recommended are E. Perret's *Navigation, Instruments, Observations, Calculs*, 1908, Paris, O. Doin et Fils, 5 francs (in *Encyclopédie scientifique*); and *Cours de Navigation et de Compas*, 1913, Ecole Navale, A. Challamel, Paris, 10 francs.

Many German texts are listed in *Katalog der Büchersammlung der Nautischen Abteilung des Reichsmarineamts*, 1905, Berlin, Mittler und Sohn.

On tables of altitudes and azimuths, see *Encyclopédie des Sciences mathématiques*, volume VII, part 1, 218-223.

MINIMUM LIST* OF BOOKS ON THE MATHEMATICAL THEORY OF BALLISTICS.

Articles in *Encyclopédie des Sciences mathématiques*, volume IV, part 6, pages 1-191.

P. R. Alger and others, *Ground Work of Practical Navy Gunnery or Exterior Ballistics*, 329 pages, 1915, Annapolis, U. S. Naval Inst., \$6.

P. Charbonnier, *Traité de Balistique*, second edition, 592 pages, 1904, Paris, Beranger; *Balistique extérieure rationnelle*, two volumes, 492 and 406 pages, 1907, Paris, Doin; *Balistique intérieure*, 351 pages, 1900, Paris, Doin.

C. Cranz, *Lehrbuch der Ballistik*, † Leipzig, Teubner; volume I, *Äussere Ballistik*, second edition, 1917, 529 pages; volume III, *Experimentelle Ballistik*, 339 pages, 1913; volume IV, *Tafeln und Photographieen*, 81 pages, 1910.

J. M. Ingalls, *Artillery Circular M, Ballistic Tables*, Revised by Ordnance Board, 233 pages, 1918, Government Printing Office, Washington.

F. Siacci, *Balistique extérieure*, 474 pages, 1892, Paris, Berger-Levrault.

E. Vallier, *Balistique extérieure*, second edition, 212 pages, Paris, Gauthier-Villars, 1913 (with improvements on Siacci's methods).

G. M. Wildrick, *Gunnery for Heavy Artillery*, Coast Artillery School, Ft. Monroe, 1918 (recent methods for effect of wind on trajectories).

* Prepared by Professors Haskins and Gronwall, with the concurrence of Professor Bliss. As supplementary books they suggest A. Hamilton, *Ballistics*, Artillery School Press, Ft. Monroe, 1908-9; J. M. Ingalls, *Interior Ballistics*, third ed., 1912, Wiley and Sons, \$3; Ingalls, *Artillery Circular N, Problems in Exterior Ballistics*, 250 pp., Government Printing Office, 1900 (essentially a revised form of his *Handbook of Problems in Direct and Indirect Fire*, 1890, Wiley & Sons); I. Didion's and N. Mayevski's texts, 1860 and 1872, of historical interest. Also *Journal of the U. S. Artillery*, Ft. Monroe, 1892-, and *U. S. Naval Institute Proceedings*, Annapolis, 1874-, each with many translations of articles in foreign artillery journals.

† English transl. by Greenhill just announced.

BOOKS ON THE THEORY OF AVIATION.*

* W. L. Cowley and H. Levy, *Aeronautics in theory and experiment*, Longmans, 1918.

A. Fage, *The Aeroplane*, London, Griffin and Company, 1917.

M. A. S. Riach, *Air screws*, New York, Appleton, 1916.

* F. Lanchester, *Aerodynamics*, 1907; *Aerodnetics*, 1908, Van Nostrand, \$6 each (classics though rather out of date).

Duchêne, *The mechanics of the aeroplane*, Longmans.

A. Klemm, *Course in aerodynamics and airplane design*, New York, Gardner, Moffat and Company, \$5 (from *Aviation and Aeronautical Engineering*, 1916-7).

F. E. Bedell, *Airplane characteristics*, Ithaca, Taylor and Company, 1918.

In addition to these particularly recommended books, access should be had to the following:

* Technical Reports of the British Advisory Committee for Aeronautics, London, yearly (Greenhill's articles, of special interest to mathematicians, have been published separately under the titles* *Dynamics of Mechanical Flight*, 1912, Van Nostrand, and *Report on Gyroscopic Theory*).

* Annual Reports of the American Advisory Committee for Aeronautics, Washington (the articles by Hunsaker, Durand and E. B. Wilson are of special interest to mathematicians).

* G. H. Bryan, *Stability in Aviation*, 1911, Macmillan, \$2.

S. Brodetsky, *Tôhoku Math. Journ.*, June, 1916; Aug., 1918.

* J. C. Hunsaker and others, *Dynamical Stability of Aeroplanes*, Smithsonian Miscellaneous Collections, 62, 1916, No. 5.

* N. Joukowski, *Bases théorétiques de l'Aéronautique: Aérodynamique*, 230 pp., 1916, Gauthier-Villars.

As regards the experimental work, recommended books are:

* G. Eiffel, *Résistance de l'air*, 1910, 5½ francs; *Resistance de l'air et l'aviation*, 1912, 12 francs (transl. by Hunsaker, *Resistance of the Air*, Houghton, Mifflin and Company, 1913); *Nouvelles recherches . . . faites au laboratoire d'Auteuil*, 1914, 50 francs.

J. C. Hunsaker, *Stable Biplane Arrangements*, *Engineering*, Jan., 1917.

* Hunsaker and others, *Reports on Wind Tunnel Experiments in Aerodynamics*, Smithsonian Inst., 1916, 30 cents.

Armand de Gramont, *Essais d'Aérodynamique*, Gauthier-Villars, 1911-4.

* A. W. Judge, *Properties of Aerofoils and Aerodynamic bodies*, 1917, Macmillan, \$6.

R. Pannell, *The wind channel*, *Aeronautical Journ.*, July, 1918.

* F. Lanchester, *Flying-machine from engineering standpoint*, 1916, Van Nostrand, \$3.

* S. P. Langley, *Experiments . . .*, Smithsonian Inst., 1891, 1908, 1911 (of historical interest only).

* L. Marchis and Riccardo-Brauzzi, in their three and two volume works each entitled *Cours d'Aéronautique*, treat the theory of balloons very fully.

I must omit the list of titles of over fifty papers of mathematical character which appeared during the past two years

* Recommended by Professor H. Bateman of the Aeronautical Laboratory of Troop College of Technology, Pasadena, Cal. The starred books are those recommended independently by Professor E. B. Wilson, who remarked that A. F. Zahm's *Aerial Navigation*, Appleton and Company, 1911, gives the best popular historical account, while the best elementary account is by Painlevé and Borel, *l'Aviation*.

in the following journals: *Aviation, Aeronautics, Aeronautical Journal, Aerial Age, Proceedings of the Royal Society of London*, volume 91A, page 354, page 503, and *Aviation and Aeronautical Engineering* (which gave two lists of books on aviation, August 15, 1916, page 31, May 1, 1917, page 306).

A PARTIAL ISOMORPH OF TRIGONOMETRY.

BY PROFESSOR E. T. BELL.

1. It is well known that the only continuous solution, $\varphi(x), \psi(x)$, of the system of functional equations

$$\begin{aligned} (1) \quad & \varphi(a+b) = \varphi(a)\varphi(b) - \psi(a)\psi(b), \\ (2) \quad & \psi(a+b) = \psi(a)\varphi(b) + \varphi(a)\psi(b), \\ (3) \quad & \varphi^2(a) + \psi^2(a) = 1 \end{aligned}$$

is $\varphi(x) \equiv \cos x, \psi(x) \equiv \sin x$. By suppressing the condition that φ, ψ , shall be continuous functions of a single variable, and one or two of (1), (2), (3), we get what may be called the partial isomorphs of trigonometry, whose interest, of course, will depend chiefly upon their interpretations. Several such are known and in use. While seeking arithmetical paraphrases for some of the more complicated results in elliptic and theta functions, I noticed incidentally another of these isomorphs in which (1), (2) only are retained. Apart from its usefulness in the theory of numbers (which is not considered here), this isomorph is of interest because it extends, in a sense, the concepts of evenness and oddness, or parity, relatively to functions of several variables.

2. We denote the sets of variables, $(x_1, x_2, \dots, x_n), (-x_1, -x_2, \dots, -x_n)$ by $\xi, -\xi$ respectively, and define a function f to be even or odd in ξ according as it does not or does change sign when the signs of $x_1, x_2, \dots, x_n$ are changed simultaneously: $f(\xi) = f(-\xi)$, or $f(\xi) = -f(-\xi)$, according as f is even or odd in ξ . These relations may be written $f((x_1, x_2, \dots, x_n)) = \pm f((-x_1, -x_2, \dots, -x_n))$, the inner $()$ being used to distinguish $f(\xi)$ from $f(x_1, x_2, \dots, x_n)$, in which no property of evenness or oddness is implied.

Two sets of variables ξ', ξ'' , are distinct if and only if they have no variable in common. Unless the contrary is expressly stated, all the sets of variables in the f 's which follow are distinct. We shall be concerned with a form of addition for sets, which is as follows: Let $\xi' \equiv (x_1', x_2', \dots, x_a')$, $\xi'' \equiv (x_1'', x_2'', \dots, x_b'')$, $\dots$, $\xi''' \equiv (x_1''', x_2''', \dots, x_c''')$; then the sum of $\xi', \xi'', \dots, \xi'''$ is the set $(\xi'; \xi''; \dots; \xi''') \equiv (x_1', x_2', \dots, x_a', x_1'', x_2'', \dots, x_b'', \dots, x_1''', x_2''', \dots, x_c''')$. Hence in particular, $\xi' \equiv (x_1'; x_2'; \dots; x_a') \equiv (x_1', x_2', \dots, x_a')$; and $(\xi'; (\xi''; \xi'''); \dots) \equiv (\xi'; \xi''; \xi'''; \dots)$ etc.; and $(\xi'; -\xi''; \xi'''; \dots) \equiv -(-\xi'; \xi''; -\xi'''; \dots)$.

3. Extending the foregoing to several sets, let $(x_{i1}, x_{i2}, x_{i3}, \dots, x_{ia_i}) \equiv \xi_i$; $(y_{j1}, y_{j2}, y_{j3}, \dots, y_{jb_j}) \equiv \eta_j$, ($i = 1, \dots, r$; $j = 1, \dots, s$), denote $(r + s)$ distinct sets of variables. Then, a function f of $\xi_1, \xi_2, \dots, \xi_r, \eta_1, \eta_2, \dots, \eta_s$ which is even in each ξ separately, odd in each η separately, is denoted by $f(\xi_1, \xi_2, \dots, \xi_r | \eta_1, \eta_2, \dots, \eta_s)$. By definition the parity of this f is $(a_1, a_2, \dots, a_r | b_1, b_2, \dots, b_s)$; its order, or total number of independent x, y variables, is $(a_1 + a_2 + \dots + a_r) + (b_1 + b_2 + \dots + b_s)$; its degree, or total number of ξ, η variables is $(r + s)$. Similarly $f(\xi_1, \xi_2, \dots, \xi_r | 0)$, $f(0 | \eta_1, \eta_2, \dots, \eta_s)$ denote f 's of respective parities $(a_1, a_2, \dots, a_r | 0)$, $(0 | b_1, b_2, \dots, b_s)$, of the obviously corresponding orders and degrees, even in each ξ , odd in each η . Beyond the prescribed conditions of evenness and oddness, all of these f 's are wholly arbitrary: they may be continuous in some or in none of the x, y variables; algebraic, or transcendental, etc. When $a_i = a$, $b_j = b$, ($i = 1, \dots, r$; $j = 1, \dots, s$), the above parities are written respectively $(a^r | b^s)$, $(a^r | 0)$, $(0 | b^s)$. The parity of a constant, i. e., function independent of the x, y variables in f , is $(0 | 0)$. According to this notation, a function of parity $(1^r | 1^s)$ is a function of $r + s$ single variables (not sets), even in r of the variables separately, odd separately in the rest; e. g.,

$$\prod_{i=1}^r \cos x_i \cdot \prod_{j=1}^s \sin y_j$$

is of parity $(1^r | 1^s)$; while

$$\cos \left(\sum_{i=1}^r x_i \right) \cdot \sin \left(\sum_{j=1}^s y_j \right)$$

is of parity $(r | s)$. As another example, an arbitrary function

of n variables can always be written as the sum of a function of parity $(n|0)$ and a function of parity $(0|n)$:

$$(4) \quad 2f(x_1, x_2, \dots, x_n) \equiv f_0((x_1, x_2, \dots, x_n)|0) \\ + f_1(0|(x_1, x_2, \dots, x_n))$$

$$(5) \quad f_0((x_1, x_2, \dots, x_n)|0) = f(x_1, x_2, \dots, x_n) \\ + f(-x_1, -x_2, \dots, -x_n)$$

$$f_1(0|(x_1, x_2, \dots, x_n)) = f(x_1, x_2, \dots, x_n) \\ - f(-x_1, -x_2, \dots, -x_n).$$

The isomorph we shall consider is the algebra of sets and parities, and the closely related properties of functions $f(\xi_1, \xi_2, \dots, \xi_r|\eta_1, \eta_2, \dots, \eta_s)$. The algebra is implicit in § 5; the correspondence with (1), (2) is established in § 6, extended in §§ 7, 8, and in §§ 9, 10 outlined for the purpose stated in § 4.

4. It is a fundamental result for the applications that an arbitrary f of parity $(a_1, a_2, \dots, a_r|b_1, b_2, \dots, b_s)$ may be expressed linearly in terms of $2^{\omega-\delta}$ suitably chosen functions whose parities are all of the type $(1^\alpha|1^\beta)$, where ω = the order of f ; δ = the degree; and $\alpha + \beta = \omega$. Although the actual linear representation is not required in use, we shall, nevertheless, show how to write it down, in proving its existence—to bring out the trigonometric analogies between $\sin x$, $\cos x$ and $(0|x)$, $(x|0)$. It will follow then that the f in (4) is a linear function of the same sort; in this case the number of functions in the representation is 2^n , and $\alpha + \beta = n$. The degrees of the several functions in the representations follow a more complicated law, which may be written down if desired. To arrive at these representations briefly, we adopt the following conventions.

5. Associated with each symbol $(\xi_1, \xi_2, \dots, \xi_r|\eta_1, \eta_2, \dots, \eta_s)$ is a symbol of parity $(a_1, a_2, \dots, a_r|b_1, b_2, \dots, b_s)$; and if $\xi_1 \equiv (\xi'; \xi''; \dots; \xi''')$, the notation being that of § 2, the parity is written $((a; b; \dots; c), a_2, \dots, a_r|b_1, b_2, \dots, b_s)$; viz., $a_1 \equiv (a; b; \dots; c)$, and so in all cases where a set ξ or η is regarded as a sum of sets. With the same notation, if $(\xi'; \xi''; \dots; \xi''')$ and $(a; b; \dots; c)$ are associates, then we take as the associate of $(-\xi'; -\xi''; \dots; \xi''')$, the parity $(-a; -b; \dots; c)$, etc., the minus signs indicating that the signs of all the variables in $\xi', \xi'', \dots$ have been changed. Hence, from the

definitions, we take $-(-a; -b; \dots; c) \equiv (a; b; \dots; -c)$, etc. For brevity we may write a symbol $(\xi_1, \xi_2, \dots, \xi_r | \eta_1, \eta_2, \dots, \eta_s) \equiv (X | Y)$, and the corresponding parity $(A | B)$. Then the formal laws of addition and multiplication for symbols $(X | Y)$, $(A | B)$ are defined as follows. An equation of the form

$$(6) \quad p(A | B) = c_1 p(A_1 | B_1) + c_2 p(A_2 | B_2) + \dots + c_k p(A_k | B_k),$$

where the $c_1, c_2, \dots, c_k$ are integers > 0 , is to be read: An arbitrary f of parity $(A | B)$ is expressible as a sum of $c_1 + c_2 + \dots + c_k$ suitably chosen functions, c_i of which are of parity $(A_i | B_i)$, ($i = 1, \dots, k$). Similarly, if some of the c 's are integers < 0 , (6) signifies the like expressibility as a difference. When we wish to consider the sets, rather than the parities, the equation (6) will be replaced by

$$(7) \quad q(X | Y) = c_1 q(X_1 | Y_1) + c_2 q(X_2 | Y_2) + \dots + c_l q(X_l | Y_l),$$

where all the c 's denote $+1$ or -1 (for an arbitrary constant, $|c| > 1$, may obviously be absorbed in the function). (7) indicates the appropriate $(X_i | Y_i)$ for the linear expression of arbitrary $f(X | Y)$; the equations (6), (7) are merely different ways of stating the same fact. The next has another meaning. If a given $F(X | Y)$ may be written in the form

$$(8) \quad cF(X | Y) = c_1 f(X_1 | Y_1) + c_2 f(X_2 | Y_2) \\ + \dots + c_k f(X_k | Y_k),$$

where the c 's are constants, by choosing the ξ, η which occur in X_i, Y_i , ($i = 1, \dots, k$), suitably in terms of the ξ, η which occur in X, Y this will be symbolized by (9), equivalent to (8),

$$(9) \quad cq(X | Y) = [c_1 q(X_1 | Y_1) + c_2 q(X_2 | Y_2) \\ + \dots + c_k q(X_k | Y_k)].$$

To define multiplication, let $t, \alpha, \beta \equiv p, A, B$, or $t, \alpha, \beta \equiv q, X, Y$; then multiplication, which is defined to be associative, commutative and distributive, for symbols $t(\alpha | \beta)$, is given by

$$(10) \quad t(\alpha_1|\beta_1)t(\alpha_2|\beta_2) \cdots t(\alpha_r|\beta_r) \\ = t(\alpha_1, \alpha_2, \cdots, \alpha_r|\beta_1, \beta_2, \cdots, \beta_r)$$

$$(11) \quad t(\alpha|\beta)\{t(\alpha_1|\beta_1) + t(\alpha_2|\beta_2)\} \\ = t(\alpha, \alpha_1|\beta, \beta_1) + t(\alpha, \alpha_2|\beta, \beta_2), \\ t(-\alpha|\beta) = t(\alpha|\beta) = -t(\alpha|-\beta).$$

The identity of addition is 0, of multiplication $(0|0)$. These laws are to be regarded as purely formal until they receive their interpretations in the addition theorems for $t(\alpha|\beta)$. One obvious interpretation of (10) is: The product of r functions on the distinct sets $(X_i|Y_i)$, $(i = 1, \cdots, r)$, is a function of parity $(A_1, A_2, \cdots, A_r|B_1, B_2, \cdots, B_r)$. Similarly for (11); but both of these are trivial. We may now write down the equations which establish the correspondence with (1), (2).

6. Let $t, \alpha, \beta \equiv p, a, b$; or $t, \alpha, \beta \equiv q, \xi', \xi''$, where $\xi' = (x_1', x_2', \cdots, x_a')$, $\xi'' = (x_1'', x_2'', \cdots, x_b'')$; then the addition theorems for t -symbols are

$$(12) \quad t((\alpha; \beta)|0) = t(\alpha|0)t(\beta|0) - t(0|\alpha)t(0|\beta), \\ \equiv t(\alpha, \beta|0) - t(0|\alpha, \beta),$$

$$(13) \quad t(0|(\alpha; \beta)) = t(0|\alpha)t(\beta|0) + t(\alpha|0)t(0|\beta), \\ \equiv t(\beta|\alpha) + t(\alpha|\beta),$$

the second forms coming from (10). Corresponding to the formulas for $\sin \alpha \cos \beta$, etc., we have

$$(14) \quad 2t(\alpha|0)t(\beta|0) \equiv 2t(\alpha, \beta|0) = [t((\alpha; -\beta)|0) + t((\alpha; \beta)|0)],$$

$$(15) \quad 2t(0|\alpha)t(0|\beta) \equiv 2t(0|\alpha, \beta) = [t((\alpha; -\beta)|0) - t((\alpha; \beta)|0)],$$

$$(16) \quad 2t(0|\alpha)t(\beta|0) \equiv 2t(\beta|\alpha) = [t(0|(\alpha; \beta)) + t(0|(\alpha; -\beta))],$$

$$(17) \quad 2t(\alpha|0)t(0|\beta) \equiv 2t(\alpha|\beta) = [t(0|(\alpha; \beta)) - t(0|(\alpha; -\beta))].$$

(16), (17) are, of course, different forms of the same thing, but it is convenient to have both.

To prove (12)–(17), we write down the identities (18) for (12), (14), (15), and (19) for (13), (16), (17),

$$f((\xi'; \xi'')|0) = f_1(\xi', \xi''|0) - f_2(0|\xi', \xi''), \\ (18) \quad 2f_1(\xi', \xi''|0) \equiv f((\xi'; -\xi'')|0) + f((\xi'; \xi'')|0), \\ 2f_2(0|\xi', \xi'') \equiv f((\xi'; -\xi'')|0) - f((\xi'; \xi'')|0).$$

$$\begin{aligned}
 f(0 | (\xi'; \xi'')) &= f_1(\xi'' | \xi') + f_2(\xi' | \xi''), \\
 (19) \quad 2f_1(\xi'' | \xi') &\equiv f(0 | (\xi'; \xi'')) + f(0 | (\xi'; -\xi'')), \\
 2f_2(\xi' | \xi'') &\equiv f(0 | (\xi'; \xi'')) - f(0 | (\xi'; -\xi'')).
 \end{aligned}$$

It only remains to show that the functions on the right of the f_1, f_2 identities have the properties implied by the notation on the left, thus, e. g.,

$$\begin{aligned}
 2f_1(-\xi'' | \xi') &\equiv f(0 | (\xi'; -\xi'')) + f(0 | (\xi'; \xi'')) = 2f_1(\xi'' | \xi'); \\
 2f_1(\xi'' | -\xi') &\equiv f(0 | (-\xi'; \xi'')) + f(0 | (-\xi'; -\xi'')) \\
 &= -f(0 | -(-\xi'; \xi'')) - f(0 | -(-\xi'; -\xi'')) \\
 &= -f(0 | (\xi'; -\xi'')) - f(0 | (\xi'; \xi'')) = -2f_1(\xi'' | \xi');
 \end{aligned}$$

and similarly the others may be verified.

7. Comparing (12), (13) with (1), (2), we see that a correspondence has been established such that if in any consequence of (1), (2) alone, when written, $\cos(a+b) = \cos a \cos b - \sin a \sin b$, $\sin(a+b) = \sin a \cos b + \cos a \sin b$, $\cos *$, $\sin *$ be replaced respectively by $t(*|0)$, $t(0|*)$, we get a result which by (12), (13) is true, and which may be translated at once into terms of functions f . Thus the formulas for the sine or cosine of n angles as a sum of products of sines and cosines of the several angles are consequences of (1), (2) alone; hence they may be translated as indicated. E. g., corresponding to $\cos(a+b+c) = \text{etc.}$; we have

$$\begin{aligned}
 t((\alpha; \beta; \gamma) | 0) &= t(\alpha | 0)t(\beta | 0)t(\gamma | 0) - t(\alpha | 0)t(0 | \beta)t(0 | \gamma) \\
 &\quad - t(\beta | 0)t(0 | \gamma)t(0 | \alpha) - t(\gamma | 0)t(0 | \alpha)t(0 | \beta) \\
 &\equiv t(\alpha, \beta, \gamma | 0) - t(\alpha | \beta, \gamma) - t(\beta | \gamma, \alpha) - t(\gamma | \alpha, \beta),
 \end{aligned}$$

which asserts that f_0, f_1, f_2, f_3 may be found such that $f((\xi_1; \xi_2; \xi_3) | 0) \equiv f_0(\xi_1, \xi_2, \xi_3 | 0) - f_1(\xi_1 | \xi_2, \xi_3) - f_2(\xi_2 | \xi_3, \xi_1) - f_3(\xi_3 | \xi_2, \xi_1)$. The actual forms of f_0, f_1, f_2, f_3 may be written down as indicated in § 9, or they follow from (14)–(17) on multiplying throughout by $t(0 | \gamma)$, $t(\gamma | 0)$ and reapplying (14) to (17) to reduce terms of the form $t(\gamma | (\alpha; \pm \beta))$, $t((\alpha; \pm \beta) | \gamma)$, $t((\alpha; \pm \beta), \gamma | 0)$, etc.; they are

$$4f_0(\xi_1, \xi_2, \xi_3|0) \equiv f + f' + f'' + f''',$$

$$4f_1(\xi_1|\xi_2, \xi_3) \equiv -f - f' + f'' + f''',$$

$$4f_2(\xi_2|\xi_3, \xi_1) \equiv -f + f' - f'' + f''',$$

$$4f_3(\xi_3|\xi_1, \xi_2) \equiv -f + f' + f'' - f''',$$

where

$$f \equiv f((\xi_1; \xi_2; \xi_3)|0),$$

$$f' \equiv f((- \xi_1; \xi_2; \xi_3)|0),$$

$$f'' \equiv f((\xi_1; - \xi_2; \xi_3)|0),$$

$$f''' \equiv f((\xi_1; \xi_2; - \xi_3)|0).$$

Similarly, corresponding to the development of $\sin(a + b + c)$,

$$\begin{aligned} t(0|(\alpha; \beta; \gamma)) &= t(\beta, \gamma|\alpha) + t(\gamma, \alpha|\beta) \\ &\quad + t(\alpha, \beta|\gamma) - t(0|\alpha, \beta, \gamma); \end{aligned}$$

$$\begin{aligned} g(0|(\xi_1; \xi_2; \xi_3)) &\equiv g_1(\xi_2, \xi_3|\xi_1) + g_2(\xi_3, \xi_1|\xi_2) \\ &\quad + g_3(\xi_1, \xi_2|\xi_3) - g_0(0|\xi_1, \xi_2, \xi_3); \end{aligned}$$

$$4g_1(\xi_2, \xi_3|\xi_1) \equiv g - g' + g'' + g''',$$

$$4g_2(\xi_3, \xi_1|\xi_2) \equiv g + g' - g'' + g''',$$

$$4g_3(\xi_1, \xi_2|\xi_3) \equiv g + g' + g'' - g''',$$

$$4g_0(0|\xi_1, \xi_2, \xi_3) \equiv -g + g' + g'' + g''',$$

where

$$g \equiv g(0|(\xi_1; \xi_2; \xi_3)),$$

$$g' \equiv g(0|(- \xi_1; \xi_2; \xi_3)),$$

$$g'' \equiv g(0|(\xi_1; - \xi_2; \xi_3)),$$

$$g''' \equiv g(0|(\xi_1; \xi_2; - \xi_3)).$$

As a last example, combining these, we will write $f(x_1, x_2, x_3)$, f arbitrary, as a linear function of 8 functions of parities $(1^\alpha|1^\beta)$, $\alpha + \beta = 3$, as stated in § 4.

$$\begin{aligned} 2f(x_1, x_2, x_3) &= F((x_1, x_2, x_3)|0) + G(0|(x_1, x_2, x_3)); \\ \begin{cases} F((x_1, x_2, x_3)|0) = f(x_1, x_2, x_3) + f(-x_1, -x_2, -x_3), \\ G(0|(x_1, x_2, x_3)) \equiv f(x_1, x_2, x_3) - f(-x_1, -x_2, -x_3); \end{cases} \end{aligned}$$

$$\left\{ \begin{aligned} F((x_1, x_2, x_3)|0) &= F_0(x_1, x_2, x_3|0) - F_1(x_1|x_2, x_3) \\ &\quad - F_2(x_2|x_1, x_3) - F_3(x_3|x_1, x_2), \\ G(0|(x_1, x_2, x_3)) &= G_1(x_2, x_3|x_1) + G_2(x_3, x_1|x_2) \\ &\quad + G_3(x_1, x_2|x_3) - G_0(0|x_1, x_2, x_3). \end{aligned} \right.$$

Whence, on applying the foregoing expansions for $t((\alpha; \beta; \gamma)|0)$, $t(0|(\alpha; \beta; \gamma))$ to F, G , we find, on putting

$$\begin{aligned} f_0 &\equiv f(x_1, x_2, x_3), & f_4 &\equiv f(x_1, -x_2, x_3), \\ f_1 &\equiv f(-x_1, -x_2, -x_3), & f_5 &\equiv f(-x_1, x_2, -x_3), \\ f_2 &\equiv f(-x_1, x_2, x_3), & f_6 &\equiv f(x_1, x_2, -x_3), \\ f_3 &\equiv f(x_1, -x_2, -x_3), & f_7 &\equiv f(-x_1, -x_2, x_3), \end{aligned}$$

the following values for $F_0, F_1, F_2, F_3, G_1, G_2, G_3, G_0$:

$$\begin{aligned} 4F_0(x_1, x_2, x_3|0) &= f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 \\ 4F_1(x_1|x_2, x_3) &= -f_0 - f_1 - f_2 - f_3 + f_4 + f_5 + f_6 + f_7 \\ 4F_2(x_2|x_3, x_1) &= -f_0 - f_1 + f_2 + f_3 - f_4 - f_5 + f_6 + f_7 \\ 4F_3(x_3|x_1, x_2) &= -f_0 - f_1 + f_2 + f_3 + f_4 + f_5 - f_6 - f_7 \\ 4G_1(x_2, x_3|x_1) &= f_0 - f_1 - f_2 + f_3 + f_4 - f_5 + f_6 - f_7 \\ 4G_2(x_3, x_1|x_2) &= f_0 - f_1 + f_2 - f_3 - f_4 + f_5 + f_6 - f_7 \\ 4G_3(x_1, x_2|x_3) &= f_0 - f_1 + f_2 - f_3 + f_4 - f_5 - f_6 + f_7 \\ 4G_0(0|x_1, x_2, x_3) &= -f_0 + f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - f_7 \end{aligned}$$

As a check, we find from these values

$$8f_0 \equiv 8f(x_1, x_2, x_3) = 4(F_0 - F_1 - F_2 - F_3 + G_1 + G_2 + G_3 - G_0).$$

It thus appears from (12), (13) that the semicolon in $(\alpha; \beta)$ plays the same part as the plus sign in $(a + b)$ in (1), (2). We next interpret the $|$ and the commas in a symbol $t((\alpha; \beta; \dots), \gamma, \dots | (\alpha'; \beta'; \dots), \delta, \dots)$.

8. To keep the writing simple, we may take the case of $t((\alpha; \beta), \gamma|0)$; the general case is treated in precisely the same way. Multiplying (12), (14), (15) throughout by $t(\gamma|0)$, we get for (12)

$$\begin{aligned} (20) \quad t((\alpha; \beta), \gamma|0) &= [t(\alpha|0)t(\beta|0) - t(0|\alpha)t(0|\beta)]t(\gamma|0), \\ &\equiv t(\alpha, \beta, \gamma|0) - t(\gamma|\alpha, \beta). \end{aligned}$$

Assuming for the moment that this is a true equation, we see that the comma in $t((\alpha; \beta), \gamma|0)$ corresponds to the $\times$ sign in $\cos(\alpha + \beta) \times \cos \gamma = [\cos \alpha \cos \beta - \sin \alpha \sin \beta] \times \cos \gamma = \text{etc.}$; and likewise in the general case for multiplication by $t(\gamma_1, \gamma_2, \dots, \gamma_r|\delta_1, \delta_2, \dots, \delta_s)$, the commas play the part of $\times$ signs. To see that (20) is a true equation, we remark that it is included in (12), (14), (15), when, considering $F((\xi'; \xi''), \xi'''|0)$ as a function of $(\xi'; \xi'')$ alone, we write $F((\xi'; \xi''), \xi'''|0) \equiv f((\xi'; \xi'')|0)$, and set in (18)

$$f_1(\xi', \xi''|0) \equiv F_1(\xi', \xi'', \xi'''|0) \quad f_2(0|\xi', \xi'') \equiv F_2(\xi'''|\xi', \xi''),$$

whence the second and third of (18) become:

$$\begin{cases} 2F_1(\xi', \xi'', \xi'''|0) \equiv F((\xi'; -\xi''), \xi'''|0) + F((\xi'; \xi''), \xi'''|0), \\ 2F_2(\xi'''|\xi', \xi'') \equiv F((\xi'; -\xi''), \xi'''|0) - F((\xi'; \xi''), \xi'''|0); \end{cases}$$

corresponding to (14), (15) multiplied by $t(\gamma|0)$:

$$\begin{cases} 2t(\alpha|0)t(\beta|0)t(\gamma|0) \equiv 2t(\alpha, \beta, \gamma|0) \\ \quad = [t((\alpha; -\beta), \gamma|0) + t((\alpha; \beta), \gamma|0)], \\ 2t(0|\alpha)t(0|\beta)t(\gamma|0) \equiv 2t(\gamma|\alpha, \beta) \\ \quad = [t((\alpha; -\beta), \gamma|0) - t((\alpha; \beta), \gamma|0)]. \end{cases}$$

Similarly, if (12) is multiplied by $t(0|\gamma)$, it is easily seen that the $|$ is a symbolic $\times$.

9. (i) Combining the correspondences established in §§ 7, 8, we have the following correspondence between t -symbols and sines and cosines. Let

$$\alpha_i \equiv (\alpha_{i1}; \alpha_{i2}; \dots; \alpha_{im_i}), \quad \beta_j \equiv (\beta_{j1}; \beta_{j2}; \dots; \beta_{jn_j}), \\ (i = 1, \dots, r; j = 1, \dots, s),$$

be any partitioning of the α_i, β_j in $t \equiv t(\alpha_1, \alpha_2, \dots, \alpha_r|\beta_1, \beta_2, \dots, \beta_s)$. Then, as the generalization of (12), (13), t may be expanded as a linear function, with coefficients ± 1 , of homogeneous products all of whose factors are $t(*|0)$'s or $t(0|\S)$'s by expanding

$$\prod_{i=1}^r \cos \left(\sum_{a=1}^{m_i} \alpha_{ia} \right) \cdot \prod_{j=1}^s \sin \left(\sum_{b=1}^{n_j} \beta_{jb} \right)$$

in the usual form as a sum of homogeneous products of $\cos *$'s $\sin \S$'s, each $*$, $\S$ representing a single α or a single β , and then replacing $\cos *$, $\sin \S$ respectively by $t(*|0)$, $t(0|\S)$. Finally, applying (10) to each term in the t -expansion, we reduce the whole to a $\Sigma \pm t(\alpha', \alpha'', \dots | \beta', \beta'', \dots)$, the α' , α'' , $\dots$, β' , β'' , $\dots$ being the α , β in the () for the partitioning of the α_i , β_j .

(ii) Similarly, the generalization of (14), viz., the expression of $2^{s-1}t(\alpha_1|0)t(\alpha_2|0) \dots t(\alpha_s|0)$ in the form $[\Sigma \pm t(\pm \alpha_1; \pm \alpha_2; \dots; \pm \alpha_s)|0]$, is written down from the expression of $2^{s-1}\cos \alpha_1 \cos \alpha_2 \dots \cos \alpha_s$ in the form $\Sigma \pm \cos(\alpha_1 \pm \alpha_2 \pm \dots \pm \alpha_s)$, a cosine term in the Σ , such as $\cos(\alpha_1 - \alpha_2 + \dots + \alpha_s)$ being replaced by $t(\alpha_1; -\alpha_2; \dots; \alpha_s)$; likewise for $c \cdot t(0|\beta_1)t(0|\beta_2) \dots t(0|\beta_s)$, where $c = (-1)^{s/2}2^{s-1}$ or $(-1)^{(s-1)/2}2^{s-1}$ according as s is even or odd, from the corresponding cosine or sine sums for $c \sin \beta_1 \sin \beta_2 \dots \sin \beta_s$. It is unnecessary to write the results: they may be obtained at once from the formulas given in books on trigonometry, e. g., E. W. Hobson's *Elementary Treatise*, §§ 49, 50 (Cambridge, 1897).

(iii) To generalize (16), (17), we multiply together the expressions found in (ii) for $t(\alpha_1|0)t(\alpha_2|0) \dots t(\alpha_r|0)$ and $t(0|\beta_1)t(0|\beta_2) \dots t(0|\beta_s)$, using (10) on each term of the distributed product. There will be four cases, according as $r, s \equiv 0, 1 \pmod{2}$. The results are rather complicated, and need not be written.

(iv) Using (i)–(iii) it is evident that $f(\xi_1, \xi_2, \dots, \xi_r | \eta_1, \eta_2, \dots, \eta_s)$ may be systematically expressed in the form stated in § 4. We have given a simple example in § 7. All of the results of this section may be proved independently by induction. This is unnecessary, however, as the formal correspondence established in §§ 7, 8 shows.

10. On account of its interest, we may conclude with the analogue of Demoivre's theorem. Since this theorem, for a positive integral exponent n , follows from (1), (2) alone, we may apply the correspondence of § 7 to get $p(n|0)$ and $p(0|n)$:

$$\{p(1|0) + ip(0|1)\}^n = p(n|0) + ip(0|n),$$

whence, equating reals and imaginaries after expansion of the left member,

$$(21) \quad \left\{ \begin{array}{l} p(n|0) = p(1^n|0) - \binom{n}{2} p(1^{n-2}|1^2) \\ \qquad \qquad \qquad + \binom{n}{4} p(1^{n-4}|1^4) - \dots, \\ p(0|n) = \binom{n}{1} p(1^{n-1}|1) - \binom{n}{3} p(1^{n-3}|1^3) \\ \qquad \qquad \qquad + \binom{n}{5} p(1^{n-5}|1^5) - \dots \end{array} \right.$$

The q -forms of these are found in the same way by distributing the product, and equating, in

$$(22) \quad \prod_{j=1}^n \{q(\xi_j|0) + iq(0|\xi_j)\} \\ = q((\xi_1, \xi_2, \dots, \xi_n)|0) + iq(0|(\xi_1, \xi_2, \dots, \xi_n)).$$

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THE TRAINS FOR THE 36 GROUPLESS TRIAD SYSTEMS ON 15 ELEMENTS.

BY PROFESSOR LOUISE D. CUMMINGS.

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In the *Transactions* for January, 1913, Professor H. S. White derived a new method for the comparison of triad systems, and applied it to the two triad systems on 13 elements. One part of a memoir published in the *Memoirs of the National Academy of Sciences*, volume 14, second memoir, shows the efficacy of this method in determining the groups, and in establishing the noncongruency of the 44 systems Δ_{15} which have a group different from the identity. The present paper gives the results of this method of comparison applied to the 36 groupless systems Δ_{15} . In this method, the triad system is regarded as an operator, and certain covariants of that operator are deduced. These covariants can be represented graphically and are called the trains of the system.

A triad system of n elements consists of triads so selected that every pair of elements occurs once, and only once, in the chosen triads. From 15 elements 455 triads can be formed. Any system contains 35 of these; leaving 420 that are called extraneous triads. Applying the system to transform them all, we see that if the system contains the 3 triads $df8, de1, fg1$, then it will transform the triad $df1$ which is not in the system and which contains the pairs $df, d1, f1$ into the triad $8eg$, but it will transform the triad $df8$ which is in the system into $df8$. Under a given triad system as an operator a triad t_1 is transformed into t_2 ; t_2 into t_3 ; t_3 into t_4 . Since only 455 triads exist, either a triad t_s of the system is reached, or else a triad that has already appeared is repeated, namely $t_{m+k} \equiv t_m$. In the former case the triad t_s recurs always, while in the latter case, the train beginning at t_m constitutes a recurring cycle. Triads that do not recur in the terminal cycle are designated as appendices, and a train consists of a recurrent cycle and all its appendices. The triad $df1$ as shown above transforms into $eg8$, similarly $eg8$ transforms into 257 which repeats indefinitely. The only other triad which transforms into 257 is 146. These 4 triads form the type of train which is exhibited graphically in the figure — $\triangleright \dashv$.

The system employed here as an operator is a Δ_{15} discovered by Professor F. N. Cole, and applied as a transformer on the 455 triads it yields the following covariants or trains: One class of trains terminating in a cycle of period 22; one class of trains terminating in a cycle of period 12; twelve classes of trains terminating in triads of the system. These trains separate the 35 triads of the system into 12 distinct classes given in the following table 1:

TABLE 1. COLE SYSTEM 36.

(1).	(2).	(3).	(4).	(5).	(6.)	(7.)	(8).	(9.)	(10).	(11).	(12).
$ab4\ bc1\ cf2\ ef4$ $af7\ bf5\ c37\ fg1$ $a18\ bg3\ dg6\ f36$ $348\ b28\ d35\ g47$ $df8$	$bd7$ $ad2$ 246 $cd4$ $c68$ $ae3$	acg $e78$ $ce5$	257	$eg2$	$de1$	167	$a56$	123	$g58$	$be6$	145

Some substitution may transform the triad system into itself. Such a substitution, evidently, must leave unchanged

the totality of trains connected with the system. Every operation of the group that leaves the system invariant must transform any train into itself or into another train of the same class. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each of the 15 elements in the 12 classes of trains terminating in triads of the system, as in the following table 2, shows the possible sets of transitive elements.

TABLE 2.

	a.	b.	c.	d.	e.	f.	g.	1.	2.	3.	4.	5.	6.	7.	8.
(1)	3	5	3	3	1	7	4	3	2	5	4	2	2	3	4
(2)	2	1	2	3	1				2	1	2		2	1	1
(3)	1		2		2		1					1		1	1
(4)									1			1		1	
(5)					1		1		1						
(6)				1	1			1							
(7)								1					1	1	
(8)	1											1	1		
(9)								1	1	1					
(10)							1					1			1
(11)		1			1								1		
(12)								1			1	1			

No two columns in table 2 are similar, therefore the system is not invariant under a single transposition, hence no substitution transforms the system into itself and the system is groupless. Similar investigations show the remaining 35 systems to be groupless.

Two systems are congruent only if their trains are identical both in type and in number. A comparison of the graphs* of the trains establishes conclusively the noncongruence of the 36 groupless systems.

The trains for the 44 systems Δ_{15} with a group furnished 216 types of covariants. This investigation shows that while many of these 216 types are found among the trains for the groupless systems, there are in addition 449 new types. Hence the 80 noncongruent systems Δ_{15} operating, as transformers, on the complete set of 455 triads formable from 15 elements, produce 665 distinct covariants. These covariants are of two kinds: (1) trains terminating in a one-term cycle, a triad of the system, and involving with the appendices from 1 to 299 triads; (2) trains ending in a terminal cycle forming a polygon

* A manuscript volume deposited in the Vassar College Library.

of 4, 6, 9, 10, 11, 12, 13, 14, 18, 20, 22, 24, 30, or 72 sides, respectively, with appendices ranging in number from 0 up to more than 100. The distinct covariants for the two noncongruent systems on 13 elements were only nine in number and much simpler in form. We see that an increase in the number n of elements, which is probably always accompanied by an increase in the number of distinct systems, produces greater complexity in the form and a very rapid increase in the number of distinct covariants connected with the noncongruent systems.

To extend this method of trains to systems on 19 or more elements would be evidently too laborious, if the object is only to classify the different triad systems. Here the analogy of invariants of algebraic forms under linear transformation is instructive; the complete calculation of systems of invariants is always possible, but only desirable when it involves finite time, as in forms of very low order. Beyond that, it is only particular forms with special invariant characters that are of general interest. So here, it is obviously most interesting to give detailed study first to triad systems which have covariant trains ending in polygon-cycles containing the largest possible number of extraneous triads. This recalls Professor E. H. Moore's study of systems whose groups are cyclic and those might probably be found again early in the proposed research.

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A THEOREM ON AREAS.

BY PROFESSOR TSURUICHI HAYASHI.

THE relative area of two given convex ovals in the same plane, swept out by moving the join of two points lying on the peripheries of the two ovals respectively, so that the point of the join dividing it in a given ratio traces out the periphery of the area containing the totality of all the points which divide the joins of two points lying on and within the two ovals respectively, satisfies the relation

$$\sqrt{S} \leq \sqrt{A} \sim \sqrt{B},$$

independent of the ratio, A , B , S being the areas of the two ovals and their relative area, respectively.

This can be proved by combining the two formulas in Elliott's paper in the *Messenger of Mathematics*, volume 7 (1878), page 151 and in Minkowski's paper in the *Mathematische Annalen*, volume 57 (1903), page 463.

But any similar formula for the relative volume of two convex ovoid bodies cannot be established.

March, 1918.

CONCERNING THE DEFINITION OF A SIMPLE CONTINUOUS ARC.

BY DR. GEORGE H. HALLETT, JR.

(Read before the American Mathematical Society October 26, 1918.)

IN a paper entitled "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, volume 33 (1911), pages 285-326, N. J. Lennes gives the following definition of a simple continuous arc.*

"A continuous simple arc connecting two points A and B , $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing A and B such that no connected proper subset of $[A]$ contains A and B ."

I shall show that the word "bounded" in this definition is superfluous.†

Lennes proves the simpler properties of formal order on an arc without any use of the assumption that it is bounded. He also proves (§§ 4, 8) that "if A_0 is any point of an arc AB , and t_1 any triangle containing A_0 as an interior point, then (in case $A_0 \neq A$) there is a point A_1 on the arc AA_0 and (in case $A_0 \neq B$) a similar point B_1 on the arc BA_0 such that every point of the arc A_1B_1 lies within t_1 ."

The following theorem also follows readily without use of the assumption that an arc is bounded:

If a point A_0 of an arc AB is a limit point of a set of points $[S]$ of the arc AB , and C is A or (if $A_0 \neq A$) any point of the

* Loc. cit., p. 308.

† Since I wrote this paper it has been pointed out to me by Professor R. L. Moore that a modification of the argument used in the proof of Theorem 49 on p. 159 of his paper "On the foundations of plane analysis situs," *Transactions Amer. Math. Society*, vol. 17 (1916), pp. 131-164, would accomplish the same result.

subarc AA_0 of AB , and D is B or (if $A_0 \neq B$) any point of the subarc A_0B of AB , then the subarc CD of AB contains at least one point of $[S]$.

This theorem is tacitly assumed by Lennes in his proof of Theorem 7 (§ 4).

All the above mentioned theorems thus hold if the term simple continuous arc is defined without the use of the word "bounded." Using this definition of an arc, I now prove the

THEOREM. *A simple continuous arc is bounded.*

Proof. Suppose AB is an arc which is not bounded. Let $[S_1]$ consist of A and all points S_1 of AB such that the arc AS_1 is bounded. Let $[S_2]$ consist of all other points of AB . By hypothesis both $[S_1]$ and $[S_2]$ exist. No point of S_2 is between A and a point of $[S_1]$. Since AB is connected, $[S_1]$ contains a limit point P_1 of $[S_2]$ or $[S_2]$ contains a limit point P_2 of $[S_1]$. In the first case any triangle t_1 containing P_1 contains an arc a_1 of AB containing P_1 . The arc a_1 contains a point Q_2 of $[S_2]$. The arc AP_1 of AB is contained in a polygon p_1 . Therefore the subarc $AQ_2 = AP_1 + P_1Q_2$ lies entirely within a polygon (Lennes, Theorem 15, § 2), and is bounded, contrary to hypothesis. In the second case any triangle t_2 about P_2 contains an arc a_2 containing P_2 , a_2 contains a point Q_1 of $[S_1]$, AQ_1 is contained in a polygon, and therefore $AP_2 = AQ_1 + Q_1P_2$ is contained in a polygon and is bounded, contrary to hypothesis. Thus in either case the supposition that AB is not bounded leads to a contradiction.

UNIVERSITY OF PENNSYLVANIA.

THE TRANSFORMATION OF A REGULAR GROUP INTO ITS CONJOINT.

BY DR. J. E. MCATEE,

(Read before the American Mathematical Society October 26, 1918.)

1. CONSIDER a regular substitution group G of order g . All the substitutions on the same letters that are commutative with every substitution of G form a group G' , of order g , called the conjoint of G . These groups are conjugate.* If G is abelian, $G = G'$. In the contrary case the statement that a

* Finite Groups, Miller, Blichfeldt and Dickson, 1916, p. 35.

substitution t of order 2 exists that transforms G into G' occurs in the literature.* Our purpose is to exhibit such a t .

Let the substitutions of G be arranged as follows:

$$G = 1, s_2, s_3, \dots, s_e, s_k, \dots, s_g,$$

where $1, s_2, s_3, \dots, s_e$ is the cyclic group generated by s_2 ($s_i = s_2^{i-1}$, $i = 2, \dots, e$). Consider the two square arrays

$$\begin{array}{cccccccc} 1, & s_2, & s_3, & \dots, & s_e, & s_k, & \dots, & s_g \\ s_2, & s_2^2, & s_3s_2, & \dots, & s_es_2, & s_ks_2, & \dots, & s_gs_2 \\ s_3, & s_2s_3, & s_3^2, & \dots, & s_es_3, & s_ks_3, & \dots, & s_gs_3 \\ . & . & . & . & . & . & . & . \\ s_e, & s_2s_e, & s_3s_e, & \dots, & s_e^2, & s_ks_e, & \dots, & s_gs_e \\ s_k, & s_2s_k, & s_3s_k, & \dots, & s_es_k, & s_k^2, & \dots, & s_gs_k \\ . & . & . & . & . & . & . & . \\ s_g, & s_2s_g, & s_3s_g, & \dots, & s_es_g, & s_ks_g, & \dots, & s_g^2 \end{array}$$

and

$$\begin{array}{cccccccc} 1, & s_2, & s_3, & \dots, & s_e, & s_k, & \dots, & s_g \\ s_2, & s_2^2, & s_2s_3, & \dots, & s_2s_e, & s_2s_k, & \dots, & s_2s_g \\ s_3, & s_3s_2, & s_3^2, & \dots, & s_3s_e, & s_3s_k, & \dots, & s_3s_g \\ . & . & . & . & . & . & . & . \\ s_e, & s_es_2, & s_es_3, & \dots, & s_e^2, & s_es_k, & \dots, & s_es_g \\ s_k, & s_ks_2, & s_ks_3, & \dots, & s_ks_e, & s_k^2, & \dots, & s_ks_g \\ . & . & . & . & . & . & . & . \\ s_g, & s_gs_2, & s_gs_3, & \dots, & s_gs_e, & s_gs_k, & \dots, & s_g^2 \end{array}$$

Each row of these squares is a permutation of the elements of the first row. Each of these permutations represents a substitution and we may assume, without loss of generality, that the substitutions thus obtained from the first square are those of G . For simplicity in what follows we shall call these substitutions $1, u_2, u_3, \dots, u_g$. The substitutions $1, u_2', u_3', \dots, u_g'$ represented by the second square are those of G' .†

The substitution represented by the row Gs_2 in the first

* Ibid., p. 46.

† Loc. cit.

square is

$$u_2 = s_1 s_2 s_3 \cdots s_e \cdot s_k s_h s_v \cdots s_i s_m s_n \cdots s_x s_q \cdots,$$

where $s_1 = 1$. The substitution represented by the row $s_2 G$ in the second square is

$$u_2' = s_1 s_2 s_3 \cdots s_e \cdot s_k s_i s_x \cdots s_h s_m s_q \cdots s_v s_n.$$

To show that the replacements indicated in u_2 and u_2' are correct we note first that the first cycles in u_2 and u_2' are the same, since s_2 is commutative with each of its powers. Also, since $s_h = s_k s_2$ and $s_i = s_2 s_k$, we have $s_2 s_h = s_i s_2$. Therefore s_i is replaced in u_2 by what s_h is in u_2' . Moreover, if s_h is replaced by s_v in u_2 , s_i by s_x in u_2' , s_x by s_q in u_2 and s_v by s_n in u_2' , it results that s_m is replaced by s_n in u_2 and by s_q in u_2' . In fact, the product $u_2 u_2'$ gives s_h replaced by s_n , while in $u_2' u_2$ we have s_h replaced by the product $s_m s_2$. Hence $s_m s_2 = s_n$. Similarly, we have that s_m is replaced by s_q in u_2' . From this we see that the substitution

$$t = (s_h s_i)(s_v s_x)(s_n s_q)$$

transforms u_2 into u_2' so far as the elements involved are concerned. Continuing this, we ultimately obtain a substitution t of order 2 that transforms any power of u_2 into the same power of u_2' . Consider next the substitutions u_j and u_j' represented by the rows $G s_j$ and $s_j G$ respectively (s_j not in the cyclic subgroup generated by s_2). In u_j we have s_2 replaced by what s_j is in u_2' and in u_j' we have s_2 replaced by what s_j is in u_2 . Now G and G' are simply isomorphic and this isomorphism may be established such that all the substitutions begin with s_2 . Then t transforms G into a group that is simply isomorphic with G' and such that the first two letters in corresponding substitutions are the same. Since G and G' are regular it follows that t transforms G into G' .

2. In the substitution u_2 of G we have the cycle

$$s_k s_k s_2 s_k s_2^2 s_k s_2^3 \cdots s_k s_2^{e-1},$$

where $s_k s_2^i$ is to be considered as one letter. Let $s_p = s_k s_2 s_k^{-1}$. Then, since $s_p^i = s_k s_2^i s_k^{-1}$, we have $s_p^i s_k = s_k s_2^i$, that is, in the substitution of G' corresponding to the row $s_p G$ will be found the above cycle. This shows that the substitutions of

G and G' are composed of the same cycles combined differently when G is not abelian. Take for example,

$$G = 1, (abc)(def), (acb)(dfe), (ad)(bf)(ce), \\ (ae)(bd)(cf), (af)(be)(cd),$$

$$G' = 1, (abc)(dfe), (acb)(def), (ad)(be)(cf), \\ (ae)(bf)(cd), (af)(bd)(ce).$$

URBANA, ILLINOIS,
September 24, 1918.

CORRECTIONS.

PROFESSOR G. LORIA has kindly pointed out the fact that the curves discussed in the first part of my article "Some Algebraic Curves" published in volume 25, pages 85–87 of the BULLETIN are special cases of curves discussed in his treatise "Spezielle Algebraische und Transcendente Ebene Kurven," volume I, pages 390–4 (1910). However the main theorem of the section, viz., the r th polar of B with respect to C_n is C_{n-r} is not found in Loria's treatise.

J. H. WEAVER.

On page 472 of the BULLETIN for July, 1918, line 10, for certain functions t read certain functions of t ; line 4 from bottom, for $t^{2\pi/p}$ read $e^{2\pi i/p}$.

On page 53 of the BULLETIN for November, 1918, line 11 from bottom, for field read fluid. On page 56, line 4, for $\tanh(\mu u)$ read $\tanh(\frac{1}{2}\mu u)$.

NOTES.

THE total membership of the American Mathematical Society on January 1, 1919, was 723, including 79 life members. The total attendance of members at all meetings held in 1918, including sectional meetings, was 222; the number of papers read was 137. The number of members attending at least one meeting was 155. Accessions to the Library in 1918 included 74 periodicals and 12 non-periodicals, making a total

of 5,561 volumes, exclusive of unbound dissertations, contained in the Library. The usual List of Officers and Members will not be issued in 1919.

At the meeting of the London mathematical society on February 13, the following papers were read: By H. S. CARSLAW, "Diffraction of waves by a wedge of any angle"; by T. C. LEWIS, "Properties of pentaspherical coordinates."

THE March number (volume 20, number 3) of the *Annals of Mathematics* contains the following papers: "On quaternions and their generalization and the history of the eight square theorem," by L. E. DICKSON; "Non-symmetric kernels of positive type," by CAROLINE E. SEELY; "Elementary properties of the Stieltjes integral," by H. E. BRAY; "A kinematical property of ruled surfaces," by J. K. WHITTEMORE; "Systems of linear inequalities," by L. L. DINES; "On the shortest line between two points in non-euclidean geometry," by T. H. GRONWALL; "The generalized gamma functions," by E. L. POST; "On the most general plane closed point-set through which it is possible to pass a simple continuous arc," by R. L. MOORE and J. R. KLINE; "Repeated integrals," by D. C. GILLESPIE.

THE following university courses in mathematics are announced for the summer session:

COLUMBIA UNIVERSITY (July 7 to August 15). By Professor JAMES MACLAY: Geometric constructions, five hours.—By Professor EDWARD KASNER: Graphical methods, including nomography, five hours; Applications of the calculus, five hours.—By Professor W. B. FITE: Theory of functions of a complex variable, five hours.

CORNELL UNIVERSITY (July 7 to August 15). By Professor VIRGIL SNYDER: Algebraic and projective geometry, five hours.—By Professor D. C. GILLESPIE: Higher analysis, five hours.—By Professor F. W. OWENS: Advanced calculus (continuation), five hours.—By Dr. H. B. OWENS: Differential equations (continuation), five hours.

UNIVERSITY OF CHICAGO (June 16 to August 29). By Professor G. A. BLISS: Differential equations, Lie theory, four

hours; Differential calculus, five hours.—By Professor H. E. SLAUGHT: Elliptic integrals, four hours; Integral calculus, five hours.—By Professor E. J. WILCZYNSKI: Metric differential geometry, four hours; Algebra, five hours.—By Professor J. W. A. YOUNG: Solid analytic geometry, four hours.—By Professor F. R. MOULTON: Solution of numerical differential equations, four hours.—By Professor W. D. MACMILLAN: Celestial mechanics, four hours.—By Professor A. B. COBLE (of the University of Illinois): Elliptic modular functions, four hours.—By Professor T. H. HILDEBRANDT (of the University of Michigan): Theory of functions of a real variable, four hours; Limits and series, five hours.—By Professor G. W. MYERS: Teaching of secondary mathematics, five hours.

PROFESSOR U. AMALDI, of the University of Modena, has been appointed professor of descriptive geometry at the University of Padua. The Italian Society of Sciences (the XL) has awarded its gold medal for 1916–1917 to Professor Amaldi for his researches in infinite groups of transformations.

PROFESSOR T. LEVI-CIVITA, of the University of Padua, has accepted a professorship of mathematical analysis at the University of Rome.

PROFESSOR C. SEVERINI, of the University of Catania, has been transferred to the University of Genoa, as professor of analysis.

DR. ELENA FREDA has been appointed docent in mathematical physics at the University of Rome.

DR. O. LAZZARINO has been appointed docent in rational mechanics at the University of Turin.

DR. L. VOLTA has been appointed docent in mathematical astronomy at the University of Genoa.

PROFESSOR G. HAMEL, of the technical school at Aix la Chapelle, has been appointed professor of mathematics at the University of Tübingen, as successor to Professor A. VON BRILL, who retires at the close of the present academic year.

DR. E. FANTA, of the German technical school at Brünn, Czeco-Slovakia, has been promoted to an associate professorship of mathematics.

DR. F. NOETHER, of the technical school at Karlsruhe, has been promoted to a professorship of mathematics and mechanics.

PROFESSOR W. GROSS, of the University of Vienna, has been awarded the Richard Lieben prize for his work in the calculus of variations.

DR. R. WEITZENBÖCK, of the University of Graz, has been appointed professor of mathematics at the German technical school at Prague.

PROFESSOR F. GRAEFE, of the technical school at Darmstadt, has retired from active teaching.

DR. H. VON SANDEN, of the University of Göttingen, has been appointed professor of mathematics and mechanics at the Clausthal school of mines.

PROFESSOR W. SCHLINK has been chosen rector of the Braunschweig technical school.

PROFESSOR STEINITZ, of the Breslau technical school, has been appointed honorary professor at the University of Breslau.

PROFESSOR J. N. VAN DER VRIES, of the University of Kansas, has resigned to continue his work, undertaken during the war, as secretary of the central district of the Chamber of Commerce of the United States at Chicago.

PROFESSOR ULISSE DINI, of the University of Pisa, died October 28, 1918, in the city of Pisa, where he was born November 14, 1845. After receiving the doctorate at the age of nineteen, he spent a year at the University of Paris, under Professors Bertrand and Hermite. In 1867 he was appointed professor of mathematics at the University of Pisa, which position he held until his death. His early memoirs

on differential geometry attracted considerable favorable attention, but he is best known by his work in analysis, particularly the "Fondamenti per la teoria delle funzioni di variabili reali," "La serie di Fourier," and the four volume treatise "Lezioni d'analisi infinitesimale," in which the foundations of the calculus are laid with greater rigor and greater generality than in any previous writings. He exercised a deep influence on the training of young mathematicians and on the organization of mathematical teaching in Italy. He was a member of the learned societies of all the countries of Europe.

PROFESSOR SIRO MEDICI, of the technical school at Florence, fell in battle October 22, 1917.

PROFESSOR M. BOTTASSO, of the University of Messina, died October 3, 1918, at the age of forty years.

PROFESSOR C. L. DOOLITTLE, of the University of Pennsylvania, director of the Flower Observatory, died March 3, 1919, at the age of seventy-five years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIEBERBACH (L.). Differential- und Integralrechnung. 2ter Band: Integralrechnung. (Leitfäden für den mathematischen und technischen Hochschulunterricht, Band 5.) Leipzig, Teubner, 1918. 144 pp. Geb. M. 3.40

BROWN (S. J.) and CAPRON (P.). Calculus. Annapolis, U. S. Naval Institute, 1918. \$3.35

CAPRON (P.). See BROWN (S. J.).

DINGELDEY (F.). See SALMON (G.).

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 1, Heft 6: M. Zacharias, Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung. Mit Zusätzen von W. F. Meyer. 2ter Teil (Schluss). Leipzig, Teubner, 1918.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 2, Heft 7: C. Segre, Mehrdimensionale Räume. Leipzig, Teubner, 1918.

FIEDLER (W.). See SALMON (G.).

- FRICKE (R.). Lehrbuch der Differential- und Integralrechnung und ihre Anwendungen. 2 Bände. Leipzig, Teubner, 1918. 8vo. 12+399+6+413 pp. Geb. M. 15.00+15.00
- GALDÉANO (Z. G. DE). Las construcciones matemáticas adaptadas al complemento de análisis infinitesimal. Zaragoza, Casanal. 8vo. 76 pp. Pes. 2.00
- . Nociones de crítica matemática. Zaragoza, Casanal. 76 pp. Pes. 2.00
- JAMES (G.). Some theorems on the summation of divergent series. (Diss., Columbia.) New York, 1917. 28 pp.
- KENYON (A. M.) and LOVITT (W. V.). Mathematics for collegiate students of agriculture and general science. Revised edition. New York, Macmillan, 1918. 12mo. 7+337 pp. \$2.00
- LOVITT (W. V.). See KENYON (A. M.).
- MEYER (W. F.). See ENCYKLOPÄDIE.
- SALMON (G.). Analytische Geometrie der Kegelschnitte. Nach der freien Bearbeitung von W. Fiedler neu herausgegeben von F. Dingeldey. 2ter Teil. 7te Auflage. Leipzig, Teubner, 1918. 8vo. 10+445 pp. Geb. M. 14.00
- SCHMEIDLER (W.). Ueber homogene kommutative Gruppen hyperkomplexer Grössen und ihre Zerlegung in unzerlegbare Faktoren. (Diss.) Göttingen, 1917.
- SCHUDEISKY (A.). Projektionslehre. (Aus Natur und Geisteswelt, Nr. 564.) Leipzig, Teubner, 1918. Geb. M. 1.50
- SEGRE (C.). See ENCYKLOPÄDIE.
- STADLER (G.). Etudes sur l'équation $\partial^2 u / \partial x^2 - a \Delta^m u = 0$. (Diss., Lund.) Lund, 1916. 6+89 pp.
- STADLER (S.). Sur les systèmes d'équations aux différences finies linéaires et homogènes. (Diss.) Lund, 1918. 4to. 61 pp.
- STURM (A.). Geschichte der Mathematik bis zum Anfange des 18. Jahrhunderts. (Sammlung Göschen, Nr. 226.) 3te Auflage. Berlin, Göschen, 1917.
- TASCHÉ (G.). Elementarer Beweis des Fermatschen Satzes. Gross-Umstadt, Selbstverlag, 1918. M. 1.00
- ZACHARIAS (M.). See ENCYKLOPÄDIE.

II. ELEMENTARY MATHEMATICS.

- AHRENS (W.). Altes und Neues aus der Unterhaltungsmathematik. Berlin, Springer, 1918. M. 5.60
- . Mathematische Spiele. (Aus Natur und Geisteswelt, Nr. 170.) 3te Auflage. Leipzig, Teubner, 1916. 114 pp. M. 1.25
- BROWN (S. J.) and CAPRON (P.). Practical algebra. Annapolis, U. S. Naval Institute, 1918. \$1.45
- CAPRON (P.). See BROWN (S. J.).
- CLARK (J. R.). See RUGG (H. O.).
- CRANTZ (P.). Ebene Trigonometrie. (Aus Natur und Geisteswelt, Nr. 431.) 2te Auflage. Leipzig, Teubner, 1918. 98 pp. Geb. M. 1.50

LENNES (N. J.). See SUTTON (C. W.).

RUGG (H. O.) and CLARK (J. R.). Fundamentals of high school mathematics. A text book designed to follow arithmetic. Experimental edition distributed at cost for experimental purposes only. Not for general commercial distribution. Yonkers-on-Hudson, World Book Company, 1918. 9+266 pp.

SUTTON (C. W.) and LENNES (N. J.). Brief business arithmetic. Boston, Allyn and Bacon, 1918. 8vo. 5+297+6 pp.

THOMSON (J. B.). The art of teaching arithmetic, a book for class teachers. London, Longmans, 1917. 8vo. 4+295 pp.

TIMERDING (H. E.). Der mathematische Unterricht in den höheren Knabenschulen nach dem Kriege. Leipzig, Teubner, 1918. M. 0.80

ZIMMERMANN (H.). Rechentafel nebst Sammlung häufig gebrauchter Zahlenwerte. 8te Auflage. Ausgabe A ohne Quadrattafel, B mit Quadrattafel. Berlin, Ernst, 1918. Geb. M. 8.00, 9.00

III. APPLIED MATHEMATICS.

BAZARD (A.). Cours de mécanique. Tome 3: Hydraulique. Nouvelle édition revue, corrigée et augmentée. Paris, A. Michel, 1918. Royal 8vo. 600 pp. Fr. 18.00

BECKER (K.). See CRANZ (C.).

BELOT (E.). L'origine des formes de la terre et des planètes. Paris, Gauthier-Villars, 1918. 8vo. 12+213 pp.

BRION (M.). Aérodynamique à l'usage des élèves pilotes. Paris, Librairie aéronautique, 1918. 8vo. Fr. 2.00

BRYANT (W. W.). Pioneers of progress. Men of science. Galileo. London, S. P. C. K., 1918. 64 pp. 2s.

CHWOLSON (O.). Lehrbuch der Physik. Band I, 1: Mechanik und Messmethoden. 2te Auflage. Herausgegeben von G. Schmid. Braunschweig, Vieweg, 1918. Geb. M. 14.40

COHN (E.). Physikalisches über Raum und Zeit. 3te Auflage. Leipzig, Teubner, 1918. M. 1.20

COMMISSAIRE (H.). Leçons de mécanique. (Classes de mathématiques A et B.) Paris, Masson, 1918. 8vo. 511 pp. Fr. 11.90

CRANZ (C.). Lehrbuch der Ballistik. Herausgegeben unter Mitwirkung K. Becker. Band 4: Sammlung von Zahlentafeln, Diagrammen, und Lichtbildern. 2te vermehrte Auflage. Leipzig, Teubner, 1918. M. 18.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band VI 1 B, Heft 4: E. v. Schweidler, Atmosphärische Elektrizität; A. Schmidt, Erdmagnetismus. Leipzig, Teubner, 1918. M. 6.40

ETCHELLS (E. F.). See MNEMONIC.

GILL (J.). Text-book of navigation and nautical astronomy, revised and enlarged by W. V. Merrifield. New edition. London, Longmans, 1918. 8vo. 8+438+12 pp.

HUMBERT (P.). Sur les surfaces de Poincaré. (Thèse de mécanique rationnelle.) Paris, Gauthier-Villars, 1918. 4to. 6+83 pp.

- HYDROGRAPHIC OFFICE.** Azimuths of the sun for latitudes extending to 70 degrees from the equator. By S. Schroeder, W. H. H. Southerland, G. W. Littlehales. 9th edition enlarged. (H. O. No. 71.) Washington, Government Printing Office, 1918. 4to. 226 pp. \$1.50
- INTERNATIONAL** catalogue of scientific literature. Fourteenth annual issue. B, Mechanics. E, Astronomy. 2 volumes. London, Harrison, 1918. 8vo. 8+134+8+175 pp. 10s. 6d. + 21s.
- KAYE (G. R.).** The astronomical observatories of Jai Singh. Calcutta, India, 1918. Folio. 8+154 pp. + 26 plates, frontispiece and maps. 14.12 rupees.
- KNESER (A.).** Mathematik und Natur. Von der Schwere. Zwei akademische Reden. Breslau, 1918. M. 1.20
- KOLLROS (L.).** Géométrie descriptive. Zürich, O. Füssli, 1918. 8vo. 8+154 pp. Fr. 5.00
- LIETZMANN (W.).** Was ist Geld? (Mathematisch-physikalische Bibliothek, Band 30.) Leipzig, Teubner, 1918. 12mo. 55 pp. M. 1.00
- LITTLEHALES (G. W.).** See **HYDROGRAPHIC OFFICE.**
- LORENZ (H.).** Ballistik. Die mechanischen Grundlagen der Lehre vom Schuss. München und Berlin, R. Oldenbourg, 1917. 8vo. 6+87 pp. Geb. M. 3.50
- LUCAS (A.).** Des phénomènes gyroscopiques et de leurs principales applications à la navigation. Paris, Challamel, 1918. 8vo. Fr. 7.80
- MARGUET (F.).** Une histoire de la navigation (1550-1750). Traduit des "Elements de navigation" de J. Robertson (1780). Paris, A. Chal-
lamel, 1918.
- MERRIFIELD (W. V.).** See **GILL (J.).**
- MNEMONIC** notation for engineering formulæ. Report of the science committee of the Concrete Institute. With explanatory notes by E. F. Etchells. London, Spon, 1918. 116 pp. 6s.
- NEUENDORFF (R.).** Praktische Mathematik. 2ter Teil: Geometrisches Zeichnen, Projektionslehre, Flächenmessung, Körpermessung. (Aus Natur und Geisteswelt, Nr. 526.) Leipzig, Teubner, 1918. 104 pp. Geb. M. 1.50
- PADERSTEIN (E.).** Die Bedeutung der nichteuklidischen Geometrie für die Grundlehren der Mechanik und ihre Entwicklung. (Diss.) Halle a. S., 1918.
- ROBERTSON (J.).** See **MARGUET (F.).**
- SCHMID (G.).** See **CHWOLSON (O.).**
- SCHMIDT (A.).** See **ENCYKLOPÄDIE.**
- SCHROEDER (S.).** See **HYDROGRAPHIC OFFICE.**
- SCHWEIDLER (E. v.).** See **ENCYKLOPÄDIE.**
- SOUTHERLAND (W. H. H.).** See **HYDROGRAPHIC OFFICE.**
- WEYL (H.).** Raum. Zeit. Materie. Vorlesungen über allgemeine Relativitätstheorie. Berlin, Springer, 1918. 234 pp. Geb. M. 14.00

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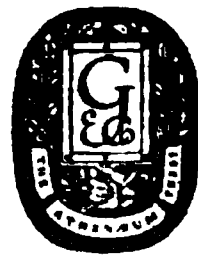
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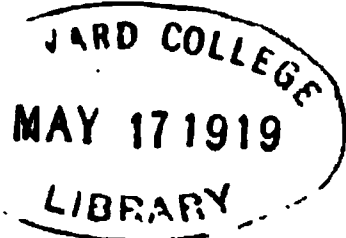


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THE LIFE AND SERVICES OF MAXIME BÔCHER.*

MAXIME BÔCHER was born in Boston, August 28, 1867, and died at his home in Cambridge, September 12, 1918. His father, Ferdinand Bôcher, was the first professor of modern languages at the Massachusetts Institute of Technology. Shortly after Mr. Charles W. Eliot, at that time professor of analytical chemistry and metallurgy in the same institution, became President of Harvard University, Professor Bôcher was called to Cambridge (in 1872) and for three decades was one of the leading teachers in the faculty of Harvard College. He was an enthusiastic collector of books. His library, which was divided after his death, formed the nucleus of the library of the French Department and yielded, furthermore, a welcome accession to the library of the Cercle Français; but its most important part, the valuable Molière and Montaigne collections, passed intact to the library of Harvard College. It was through the generosity of Mr. James Hazen Hyde, who bought the whole library, that such a disposition of the books became possible.

As a college teacher Ferdinand Bôcher is remembered by many men for whom college life in their student days offered varied attractions, as one who helped them to see and enjoy the beauty of language and literature.

Maxime's mother was Caroline Little, of Boston. She was of Pilgrim ancestry, being a descendant of Thomas Little, who joined the Plymouth Colony in its early days and in 1633 married Anne Warren, the daughter of Richard Warren, who came in the Mayflower.

Thus Bôcher's boyhood was passed in a home in which much that is best in the spirit and thought of France was united with the traditions and intellectual life of New England. He attended various schools, both public and private, in Boston and Cambridge; but it was to the influence of his parents that the awakening of his interest in science was due.

He graduated at the Cambridge Latin School in 1883 and took the bachelor's degree at Harvard in 1888. Then followed three years of study at Göttingen, where he received the degree

* For a critical estimate of Bôcher's scientific work the reader is referred to Professor Birkhoff's article in the February BULLETIN.

of doctor of philosophy in 1891, and at the same time the prize offered in mathematics by the philosophical faculty of the university. From 1891 till his death he was a member of the Department of Mathematics in Harvard University. He married Miss Marie Niemann, of Göttingen, in 1891. His wife and three children, Helen, Esther, and Frederick, survive him.

His college course was a broad one. Outside of his main field of mathematics and the neighboring field of physics he took a course in Latin and two courses in chemistry, and courses in philosophy under Professor Palmer, in zoology under Professor Mark, and in physical geography and meteorology under Professor Davis; and it is interesting to note that in his senior year, beside his work in mathematics, he elected Professor Norton's course in Roman and mediæval art; a course in music with Professor Paine, and an advanced course in geology with Professor Shaler and Professor Davis, and Professor (then Mr.) Wolff. In his senior year he also competed for a Bowdoin Prize, and the committee awarded him a second prize for an essay on "The meteorological labors of Dove, Redfield, and Espy." At graduation he received the bachelor's degree *summa cum laude*, with highest honors in mathematics, his thesis being "On three systems of parabolic coordinates." A travelling fellowship was granted him, and it was twice renewed.

Bôcher's education was not confined to the courses he took. He was a reader and a thinker, and he was interested in many of the general questions of the day. But generalities did not satisfy him; he demanded of himself that he know precisely the essential facts. His critical powers were early cultivated, and he was endowed with good judgment. In debate, he was able to marshal his facts with rapidity, to arrange them strategically, and to make his point with clearness. In rebuttal, he was an expert.

I recall an incident which occurred at a meeting of the M. P. Club* in the early nineties, and which shows the characteristics last mentioned. Professor Woods had given an interesting talk on surfaces which are applicable to one another, and had illustrated his subject with models from the Brill collection in the mathematical library of the Institute.

* This club was formed in the eighties for the purpose of bringing the members of the departments of mathematics and physics at the Massachusetts Institute of Technology and Harvard into closer relations.

One of the members of the Club was a physician, whose interest in mathematics had been kindled by Benjamin Peirce, and who, though not a profound mathematician, nevertheless delighted to read mathematics, much as our ancestors read their Horace. He asked a question which was based on his doubt whether parallel lines, in any logically necessary interpretation of the words, "meet at infinity." Now, there was also present a learned professor from another institution, and it pleased him to answer the doctor from a mighty height. But, in his answer, he was thinking only of projective geometry, and his arrogance made Bôcher indignant. "That is not true in the geometry of inversion," the latter replied. "That is not geometry," was the professor's scornful rejoinder. "It is what Klein calls geometry," came back quick as a flash. "Oh, Klein is not a geometer." This was the professor's last shot. In two brief statements of facts the youthful Bôcher had put his opponent into the position of asserting that the man who wrote the "*Vergleichende Betrachtungen über neuere geometrische Forschungen*" was not a geometer!

Above all, Bôcher was sincere. He liked to argue and to defend a position; but when the game was over, it was the truth which had been brought out that pleased him most.

He distrusted popular conclusions, even when the public was a learned one. It was facts, not views, that he sought, and his own intellect was the final arbiter. The following incident is characteristic of his type of mind. When his last sickness was developing, he needed a physician, and the well-known doctors were away in the war. He made inquiries one day regarding a young practitioner of rising fame, with whom Professor Birkhoff had recently had some experience. The latter said in closing, "I must add, however, that Dr. — is pessimistic. He is given to taking a gloomy view of the condition of his patients." "I do not care whether he is pessimistic or not," was Bôcher's reply, "if the diagnosis is correct."

The later years of his life were not happy ones. Even as far back as the winter of 1913-14 his strength was frequently inadequate for the daily needs. He never complained; in fact, he was unwilling to talk about himself even for a moment. But for one whose demands on himself were such as Bôcher's it must have been a severe trial not to achieve the full measure of results of which the mind was capable and for which it longed to work.

He was a Puritan, and with the virtues he had also the faults of the Puritan. There was no place in his world for human weakness, even though the individual had done his best. A reverence for human beings because of their struggles to attain higher things was lacking in his make-up; he respected only results. And so, to many a man who came into personal relations with him in his profession, he seemed cold and unsympathetic. What the stranger, however, too often failed to observe was that Bôcher applied the same stern standards to himself. Why should others expect to fare better?

In order to understand the mathematical work of Bôcher it is well to consider at the outset the state of the science as he found it. The nineteenth century was an era of intense mathematical activity, not in one land alone, but among all the peoples which were leaders in scientific thought. If it was not reserved for mathematicians to make formal discoveries coordinate in importance with those which formed the crown of the discoverers and early developers of the calculus, it is none the less true that mathematical imagination never played more freely, not only in geometry and algebra, but also in analysis and mathematical physics.

But mathematics was no longer in its infancy. In the great age just preceding the French revolution, a mathematician could know, at least in its essential parts, all that had been done in the science up to that time, just as, a century earlier, the man of learning was conversant not only with mathematics and physics, but also with the principal systems of philosophy. With the enormous expansion of the subject matter, or detailed theories, which grew up and flourished with amazing virility in an age characterized by its struggle for intellectual freedom, a point had been reached where it seemed as if mathematics was destined to disintegrate through the very volume of its scientific content.

It was at this time—the eve of the Franco-Prussian War—that two youths met in Berlin, who were to become leaders in mathematical thought—Felix Klein and Sophus Lie. True, these men were remote from each other in their specific mathematical interests, and their life work lay in different fields. Lie built up a consistent, harmonious theory which both yielded new results and brought old ones under a common point of view. With Klein it was not a question of developing

a method for its own sake, or even of caring for method, except in so far as he was thus able to uncover the natural interrelations of parts of the science which hitherto had seemed foreign to each other.

A pupil of Clebsch and Plücker, Klein early became acquainted with the geometric advances that group themselves about the names of Monge and Poncelet, of Steiner and von Staudt. In analysis, the theory of functions, as developed by Cauchy and his followers, was already beginning to come into its own. Göttingen was filled with the traditions of Riemann, whose life touched fingers with that of Klein. In algebra, Galois's contributions were still new, and Hermite and Kronecker had, hardly more than a decade previously, solved the general equation of the fifth degree.

Klein's first great contribution toward unifying apparently unrelated disciplines was the Erlanger Programm of 1872 mentioned above, on a Comparative Consideration of Recent Advances in Geometry. It was here that he set forth projective geometry, not as an isolated science—geometry, par excellence—but rather as one (true, the most important) of a whole array of geometries, of which, in particular, the geometry of reciprocal radii, or inversion, is a member; for the basis of each of the geometries is the group of transformations which leave invariant certain configurations, and two geometries are essentially equivalent when their groups are isomorphic and their elements stand to each other in a one-to-one and continuous relation.

It would have been easy for Klein at this stage to found a school on the basis of postulates. If the thought ever occurred to him, he rejected it both because the results to be obtained would lack important mathematical content and because he instinctively sought the specific interrelations of seemingly distinct branches of mathematics, in order that one might yield new theorems, or illumine old ones, in the other.

It was to an environment imbued with such traditions that Bôcher came, when he was matriculated at Göttingen in the fall of 1888. His previous training at Harvard had prepared him to enter at once on advanced work. In the last year of his college course, as has already been said, he had written a thesis on parabolic coordinates. Klein was beginning the continuation of his lectures on the potential function,

and these were followed by his lectures on the partial differential equations of mathematical physics, and on the functions of Lamé. He also lectured at this time on non-euclidean geometry.

It is seldom that a student is brought into such vital contact with the chief branches of mathematics as was the case with Bôcher. His thesis was on Developments of the Potential Function into Series, a subject which he shortly after worked out at greater length in a monograph. Though the leading ideas had been set forth by Klein in his lectures, nothing could be further from the truth than to think that Bôcher merely elaborated some details. The subject was an exceedingly broad one. It required for its treatment not so much a specific knowledge of the theory of the potential, although Bôcher was thoroughly equipped on that side; nor even familiarity with the geometry of inversion, of which he had made himself master; but rather, the power to carry through a piece of detailed analytic investigation with accuracy and skill, and with this work Klein occupied himself only in the most general way. Nor was it a question of the proof of convergence for the series obtained. Indeed, these proofs have not as yet been given, though recent advances have been made by Hilbert with the aid of integral equations. The importance of the dissertation in its influence on Bôcher lay largely in the fact that it stimulated his interest in mathematical physics, in pure geometry, in algebra, and in applied analysis. More precisely, beside the general geometrical ideas and theories above mentioned, the specific study of the Dupin cyclides and their generalization by Laguerre, Moutard, and Darboux was involved.* Through the method of elementary divisors, he was led to examine in detail a chapter in pure algebra, together with its application in more than a single field in geometry. From the formal solution of the first boundary value problem for Laplace's equation by means of series to the study of boundary value problems for the partial differential equations of physics of other than the elliptic type and the treatment of these problems by the more recently developed methods of integral equations, was a natural course. Throughout all his work, the total linear homogeneous differential equations of the second order were a constant source of further investiga-

* An appreciation of Bôcher's contributions in this field will be found in Professor Birkhoff's article.

tions, both by himself and by his pupils, and his last great published work, the Paris lectures, is in this field.

In the fall of 1891 Bôcher began his career as university teacher, being appointed to an instructorship in the department of mathematics of Harvard University. It is the practice of that department to give to each of its members an elementary, an intermediate, and an advanced course. Bôcher's teaching, both elementary and advanced, was successful from the beginning. He did not have to "learn to teach"; teaching came to him naturally. Doubtless he was aided in this direction by the example of his father and the family traditions, for his mother had also been a teacher; nor were his parents the only ones of the immediate family who had been engaged in that profession. The standards of clearness, both in thought and expression, which characterize French men of letters and science, Bôcher made his own, not by a conscious effort, but through an inner driving force which made it a part of his very nature to find suitable expression for his ideas. "He never tried to be clear," Major Julian Coolidge wrote me this fall, "because his constitution was such that he did not know how to express his thoughts in any but the clearest form." I would not, however, be understood as saying that he achieved his success as a teacher without effort. He gave careful thought to the preparation of all his instruction, both as regards the choice of material and the presentation; but he was able to do this without serious loss of time or energy.

His intermediate course in the first year of his teaching was on modern geometry. Professor Byerly had already developed this course to a point which gave it an important place in the undergraduate instruction. The outlook on geometry which Bôcher had acquired under Klein enabled him to make the course still more effective as an introduction to the ideas and methods of the higher geometry of the present day. He gave this course repeatedly (about every other year) during the whole period of his service, and he was engaged in the preparation of his lectures for publication at the time of his last illness.*

* Since this course meant so much to Professor Bôcher, the reader will be interested in his description of it in the Departmental Pamphlet:

"3. INTRODUCTION TO MODERN GEOMETRY AND MODERN ALGEBRA.

The subjects considered in this course are:

(a) Affine Transformations; The use of Imaginaries in Geometry;

In the minds of some readers the word *lecture* in connection with a sophomore course may cause doubts as to the efficiency of the instruction. The objection is raised that sophomores cannot take notes and get only vague outlines of ideas which they cannot develop further. It must be remembered, however, that this course is a free elective, and that it is chosen by men who have interest in mathematics and capacity for its pursuit. Moreover, frequent exercises are assigned (as a rule, daily) which range all the way from simple tests on the essentials to problems whose solution is possible only for students who really dominate the methods. These problems are corrected and returned to the student.

So much by way of apology. Let me now say, with all aggressiveness, that it was largely to the lecture method that both Professor Bôcher and I owed the awakening of our interest in mathematics when we were undergraduates in Harvard College. The instruction thus imparted stimulated thought, and the exercises assigned developed power—the power to obtain new results, even in the undergraduate stage. It was with exultation that we followed courses given by this method, in which our mathematical powers grew before our very eyes. In saying this, I am also stating Bôcher's views, for he repeatedly expressed himself on this subject in conversation.

Bôcher's advanced course in the first year of his professional life took the form of a seminary, the subject being curvilinear coordinates and functions defined by differential equations. A part of the instruction consisted of formal lectures on the latter topic, and he thus began, even at that early date, to treat topics in a field of analysis in which he was to become eminent.

In the eighties, a number of American students of mathematics from various colleges went abroad, chiefly to Germany, for further instruction and guidance in mathematics. When

Abridged Notation; Homogeneous Coordinates; Intersection and Contact of Conics; Envelopes; Reciprocal Polars; The Parametric Representation of Straight Lines and Conics; Cross-Ratio; Projection and Collineation; Inversion.

(b) Complex Quantities; The Elements of the Theory of Equations; Determinants; The Fundamental Conceptions in the Theory of Invariants.

The portion of the course devoted to the geometrical subjects (a) will be two or three times as extensive as that devoted to the algebraical subjects (b), and the relations between these two parts of the course will be emphasized."

they returned, some of them became university teachers and strove, so far as in them lay, to give to their students advantages like those to be found in Europe at a mathematical center. Bôcher was one of this latter group. With rare discernment for problems of importance, on which advanced students might work with a reasonable prospect of success, he gave himself unstintingly to the task of helping such students to carry through pieces of investigation and to put their results into good form. He did not foster work on the part of his students by artificial means—by high praise or an appeal to ambition. He felt that the student must be possessed of idealism and must, of his own nature, find satisfaction in scientific activity; otherwise, the writing of a doctor's thesis would represent only a forced growth. At times, it seemed to the beginner in research that he was unappreciative. But the student who had capacity for mathematical investigation and loved the science found an open ear and a ready response when he came with a contribution of real scientific merit, be that contribution in itself large or small.

The awakening in the science of mathematics in this country was followed at once by the springing up of the New York Mathematical Society, which shortly after became the American Mathematical Society. Of the latter Bôcher early became a member, and he took a keen interest in its affairs, contributing to its *BULLETIN* and participating in all its activities. He and Professor Pierpont were the speakers at the first Colloquium given by its members—at Buffalo, in 1896. When the establishment by the Society of a journal devoted to research was under discussion, it was through his insight that a way out of the difficulties which seemed insurmountable was found. Among the older members of the Society were those who saw in the establishment of such a publication an unfriendly act toward the *American Journal of Mathematics*. At a meeting of about a dozen mathematicians, held in New York in the fall of 1898 to discuss the question, this view was represented by the late Dr. McClintock. Bôcher asked him if he saw any objection to the Society's publishing its *Transactions*. To the surprise of all, there came a prompt answer in the negative. The difficulty was overcome. The Society might not establish the "Journal of the American Mathematical Society," but it might publish the "Transactions of the American Mathematical Society"!

Producers of mathematical research are, as a rule, not facile in their expression. When one has been engaged in the protracted study of a problem, the early difficulties and the underlying ideas become obscured through familiarity with the facts, and the writer produces a paper hard to read. The early editors of the *Transactions* labored, and not without success, to impress on the contributors the importance of making easily accessible to the reader the main results and methods of the paper, and of showing the relation of the investigation to previous work. It was here that Bôcher's power as a critic was of great service. But a critic, to be helpful in such work, must be constructive. How admirably Bôcher was adapted for this undertaking, could not be shown more strikingly than by the opening paragraphs of his Dissertation, which are a model of what an introduction to a scientific paper of wide scope should be. He was not a member of the first editorial boards, for at that time the *Annals of Mathematics* had just been taken over by Harvard University, and he was doing similar work for that journal. But from the start he was in close touch with the editors of the *Transactions*, and his views on general questions and specific papers were helpful to them. Later, he served for two terms (with the exception of one year, in which he was absent from the country) on the editorial board.

He was president of the Society from 1908 to 1910. For his presidential address he took as the subject: "The published and unpublished works of Charles Sturm on algebraic and differential equations." He delivered the address in Chicago. The meeting will live in the memories of all who were present, especially in those of the eastern colleagues, as a particularly delightful occasion.

Bôcher's judgment of men, too, was sound, and those who had occasion to discuss nominations or appointments with him felt that a decision which had his approval could be trusted.

The breadth of Bôcher's knowledge of mathematics was accompanied by a true sense of perspective. His estimate of the importance of an investigation was extraordinarily sound. In his own work, this quality of mind was both a help and a hindrance. It helped him to choose well the problems which he and his students were to study. It can fairly be said that Bôcher never occupied himself with an unimportant problem.

On the other hand, the enthusiasm just of doing things in mathematics—the joy of living, so to speak—gives to one's mental work a momentum which carries it over the obstacles of disappointment and discouragement, when one effort and another fail to yield results, and along with much which is valueless for others there come, now and then, contributions worthy of a lasting place in the science. I will not say that Bôcher was without such enthusiasm; but he did not show it in his intercourse with others. His nature was reserved. He would not talk on personal matters relating to himself, and this disinclination extended even to his scientific work.

He was, however, glad to discuss the work of others with them. He was quick to grasp the central idea and often could express it more clearly than its author. The early meetings of the Society were prized by those who attended them less for the formal papers presented than for the informal gatherings in the evening or about the breakfast table. It was here that the real mathematical discussions took place, and who of those who had the rare good fortune to be associated with that little group will ever forget what Bôcher was to us in those days? His special field was analysis; but so broad were his sympathies and his learning that he usually took a leading part in the discussions. His criticism was always helpful, often constructive, and freely given in the finest spirit.

We have mentioned the Presidential Address. At the St. Louis Congress, in 1904, he delivered an address on "The fundamental conceptions and methods of mathematics." He gave a lecture at the Fifth International Congress of Mathematicians, at Cambridge, England, in 1912, his subject being: "Boundary problems in one dimension." In 1913–14 he was exchange professor at Paris. His opening lecture was of a general nature and was entitled: "Charles Sturm et les mathématiques modernes."

It was not until late that Bôcher occupied himself with the writing of text-books. He had published some expository articles, chief among which were the pamphlet on "Regular points of linear differential equations of the second order," Harvard University Press, 1896; an article on "The theory of linear dependence," *Annals of Mathematics*, (2) 2 (1901) and an "Introduction to the theory of Fourier's series," *ibid.*, (2) 7 (1906); three years later he wrote Tract 10 of the Cambridge (England) series, entitled: "An introduction to the study of integral equations."

The Algebra appeared in 1907. Hitherto, books on algebra in the English language had been of the Todhunter type, or they had followed the lead of Salmon, through whom "Higher Algebra" came to mean specifically the study of the algebraic invariants of a linear transformation. What the mathematician needed to know of linear dependence and the theory of linear equations, of polynomials (factorization, resultants, and discriminants), the reduction of one or of two quadratic forms to normal type (including, perhaps, the rudiments of elementary divisors) he had to pick up as best he could. In no one place were they treated systematically, and most of the treatments were inadequate for the present day needs of the science.

Bôcher filled this gap in a thoroughly satisfactory manner. The Algebra was received with appreciation, both in this country and abroad, and at the suggestion of Professor Study a German translation was prepared. How thoroughly the work had been done originally is seen from the fact that practically no revision was needed.

The Trigonometry (written jointly with Mr. Gaylord) and the Analytic Geometry are so widely and intimately known as to require no detailed comment. These books present elementary subjects in a form accessible for elementary students, and treat them with a degree of accuracy, elegance, and perspective seldom attained by writers of text-books.

I have spoken of Klein's efforts to *unify* mathematics. Bôcher's aim may be described by saying that he strove to *clarify* mathematics. To illustrate by a single, but important example, let me consider the theory of functions of a complex variable. In the early nineties there were two distinct schools, and neither sought to aid or to learn from the other. Cauchy based his theory on the calculus of residues, obtaining Taylor's theorem as a corollary. With Weierstrass and Méray power series formed the foundation. The integral was more pliable and better adapted to the needs of the subject. But the questions which the critics had raised regarding limits, and in particular the reversal of the order in a double limit, had not been settled in a satisfactory manner for integrals, and even for series they were ignored by the writers of the Cauchy school. On the other hand, Weierstrass restricted his infinite processes to differentiation and power series. His treatment was rigorous, but clumsy, and the whole theory took on a formidable aspect.

Riemann's methods were thought of less as forming an independent theory than as yielding an important mode of treatment for certain classes of functions; e. g., the algebraic functions and their integrals, and the functions defined by linear differential equations or their resolvents; notably, the *P*-function and the automorphic functions.

In 1893-94 Bôcher gave for the first time in his career the introductory course on the theory of functions of a complex variable, and in the same year he repeated his course on functions defined by differential equations, laying stress on the complex theory. The subjective effect is obvious. For him, it could not be a question of developing the general theory of functions as an end in itself. He was interested in the theory as a tool—as a means of investigating, for example, the functions defined by differential equations. But he was interested in improving the tool, in developing better machinery than had come down to us. He cared nothing for the schools. He sought the simplest method for solving each problem.

Of course, he was rigorous. But for him, rigor was not a strait-jacket. For him, rigor was not something superimposed on a proof, already satisfactory to a normal mind, by a certain cult of mathematicians. If a proof was not rigorous, it was not *clear*—it had not succeeded in analyzing completely the situation. Not that, with him, there was no place for intuition in mathematics. Quite the reverse. He recognized clearly that rigor is relative, depending on the domain of conceptions and the logical maturity of the student, and he was a master of diagnosis in determining what his students required or could receive, and what their minds must reject.

His contributions of the kind we have been considering were not confined to improving proofs already complete. He discovered gaps and filled them; as in the case of the theorem that a function which is harmonic in the neighborhood of a point, that point excepted, and becomes infinite there, must be of the form (when $n = 2$):

$$u = k \log r + \omega,$$

where ω is harmonic at the point, also.

How extensive and how useful this work of Bôcher's was will become evident to any one who will turn to the writer's *Funktionentheorie*, volume I, and look up the references under Bôcher's name in the index. And what he did in this field,

he did in others. His Algebra, for example, affords numberless instances in point.

In the early years of our professional lives we were in constant intercourse over such matters. Each of us was seeking to clarify and simplify his subject. Neither of us regarded the theory of functions of a real or of a complex variable as an end in itself, for each had his own ulterior uses for the theory—Bôcher, his differential equations, both complex and real. In fact, for each of us the theory of functions was *applied mathematics*, and in presenting its subject matter and its methods to our students, our aim was to show them great problems of analysis, of geometry, and of mathematical physics which can be solved by the aid of that theory.

Bôcher was quick to grasp the large ideas of the mathematics that unfolded itself before our eyes in those early years. His attitude toward mathematics helped me to have the courage of my convictions. The *Funktionentheorie* is largely Bôcher's work, less through the specific contributions cited on its pages than through the influence he had exerted prior to 1897—long before a line of the book had been written. We worked together, not as collaborators, but as those who hold the same ideals and try to attain them by the same methods. It was constructive work, and in such Bôcher was ever eager to engage.

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December, 1918.

A THEOREM ON LINEAR POINT SETS.

BY DR. HENRY BLUMBERG.

(Read before the American Mathematical Society December 28, 1918.)

LET A be any given linear point set. We define the "relative exterior measure*" of A in the interval I as $m_e(A, I)/l$, where $m_e(A, I)$ represents the exterior measure (Lebesgue) of the subset of A in I , and l is the length of I . We then define the "relative exterior measure of A at the point x " as

* Cf. Denjoy, *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 130.

k , if

$$\lim_{l_n \rightarrow 0} \frac{m_e(A, I_n)}{l_n} = k$$

for every sequence $\{I_n\}$ of intervals enclosing x and having a length l_n that approaches 0 with increasing n . It is the purpose of this note to prove the

THEOREM. *The relative exterior measure of every linear point set exists and is equal to 1 at every one of its points, with the possible exception of those of a set of measure 0.*

Proof. Let M represent the subset of the given set A that consists of points where the relative exterior measure of A is not 1. Hence there exists, for every point x of M , an integer n_x such that for every $\epsilon > 0$ there is an interval enclosing x and of length $< \epsilon$ in which the relative exterior measure of A is $< 1 - 1/n_x$. For a given x , let n_x' be the smallest integer having the property just described. Designating by M_n the set of x 's for which $n_x' = n$, we obtain the decomposition

$$M = M_2 + M_3 + \cdots + M_n + \cdots$$

Our theorem will be established if we show that M_n is of measure zero.

Let I , of length l , be any interval of the linear continuum. According to the definition of M_n , every point x of M_n may be enclosed in an arbitrarily small interval J_x in which the relative exterior measure of M and a fortiori of M_n is $< 1 - 1/n$. The intervals J_x may be so chosen that they lie entirely in an interval I' of length $< l + \delta$, where δ is any given positive number. According to a well known theorem, we may select from the J_x 's a denumerable infinity $[J', J'', J''', \dots]$ having the same interior points as all the J_x 's. Let m_n be the measure of the portion of I' covered by the intervals $J', J'', \dots, J^{(n)}$. We distinguish two possibilities:

$$(1) \quad \lim_{n \rightarrow \infty} m_n \leq \frac{2}{3}l$$

and

$$(2) \quad \lim_{n \rightarrow \infty} m_n > \frac{2}{3}l.$$

We shall prove that in either case the exterior measure of the subset of M_n in I is $\leq [1 - (1/3n)]l$. In the first case, this is

evident. In the second case, suppose $m_n > \frac{2}{3}l$. It is easily shown by elementary reasoning that we may extract from the sequence $J', J'', J''', \dots, J^{(\nu)}$ a sequence $\bar{J}', \bar{J}'', \dots, \bar{J}^{(\bar{\nu})}$ which cover the same portion of I' and no part more than twice. Since the relative exterior measure of M_n in every J is $< 1 - 1/n$, it follows that the interior measure of the set complementary to M_n is greater than $\bar{l}_\sigma/n$ in every $\bar{J}_\sigma$ (of length $\bar{l}_\sigma$); and since no part of the continuum is covered by more than two $\bar{J}$'s, we conclude that the total interior measure of this complementary set in the portion covered by the $\bar{J}_\sigma$ [$\sigma = 1, 2, \dots, \bar{\nu}$], is $> \frac{1}{2} \cdot (1/n) \cdot \frac{2}{3}l = l/3n$. Therefore the exterior measure of the subset of M_n in I is $< l + \delta - (l/3n)$; and since δ may be chosen arbitrarily small, this exterior measure is $\leq l - (l/3n)$.

Having thus proved that the relative exterior measure of M_n is $\leq 1 - (1/3n)$ in every interval, we may now show that the measure of M_n is zero. To this end, we show that *a linear set S whose relative exterior measure is $< 1 - k$, $k > 0$, in every interval is necessarily of measure zero*. For let m = exterior measure of S . For every given positive ϵ we may then enclose S in a set of intervals I_n of total length $< m + \epsilon$. Furthermore, the subset of S in each I_n may be enclosed in a set of intervals of total length $< (1 - k)l_n$, where l_n = length of I_n ; and therefore the entire set S , in a set of intervals of total length $< \sum_n (1 - k)l_n \leq (1 - k)(m + \epsilon)$. Therefore

$$(1 - k)(m + \epsilon) > m, \quad m < \frac{(1 - k)\epsilon}{k},$$

and accordingly $m = 0$. Our theorem is thus proved.

Let Z be the subset of A of zero measure at the points of which the relative exterior measure of A is not 1; and let $H = A - Z$ be the remaining set. Since A and H differ by a set of zero measure, the relative exterior measure of the one is the same at every point as that of the other. Therefore H has the relative measure 1 at every one of its points, and may be thought of as "homogeneous" as to exterior measure. We thus have

COROLLARY 1. *Every linear point set A may be represented as*

$$A = H + Z,$$

where H is a ("homogeneous") set having relative exterior measure 1 at every one of its points, and Z is of measure zero.

We obtain a particular case of our theorem if we assume A to be a measurable set. Exterior measure will then be replaced by measure, and relative exterior measure by "relative measure." We thus have

COROLLARY 2. *The relative measure of a measurable set is 1 at every one of its points except possibly at those of a set of measure zero.*

Corollary 2 is equivalent to a theorem of Lebesgue-Denjoy.* The present note, therefore, also gives a very simple proof of this important theorem.

So far the author has not succeeded in proving the theorem of this note for higher dimensions, although there seems to be little ground for doubting its validity in n -space.

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A GENERAL FORM OF GREEN'S THEOREM.

BY PROFESSOR P. J. DANIELL.

In this paper a form of Green's theorem is considered which applies, on the one hand, to the boundary of any set E , measurable Borel, and relates, on the other hand, to potential functions which satisfy a general integral form of Poisson's equation,

$$\int_{B(E)} \frac{\partial V}{\partial n} ds = \int_E d\alpha(x, y),$$

where $\alpha(x, y)$ is some function of limited variation in (x, y) . In particular it can be used in mathematical physics in problems in which mass (or electric charge) is not distributed continuously.

Let $V_1(x, y)$, $V_2(x, y)$ be two potential functions defined and

* Lebesgue, *Leçons sur l'Intégration*, pp. 123-124, and Denjoy, loc. cit., pp. 132-137. "Relative measure" is equivalent with Denjoy's "épaisseur." Lebesgue's considerations are indirect (as far as the theorem in question is concerned), being based on properties of integrals. Denjoy's proof is direct, but still comparatively involved and long.

"differentiable"* in the fundamental square $J(0 \leq x \leq 1, 0 \leq y \leq 1)$; let

$$\frac{\partial V_1}{\partial x} = v_1, \quad \frac{\partial V_1}{\partial y} = -u_1, \quad \frac{\partial V_2}{\partial x} = v_2, \quad \frac{\partial V_2}{\partial y} = -u_2$$

be summable with their squares in J .

Furthermore assume that u_1, u_2 satisfy R_1 ; v_1, v_2 satisfy R_2 .
 R_1 . The total variation of $u(x, y)$ varying $y(0 \leq y \leq 1)$ is a function of x , finite nearly everywhere and summable in $(0 \leq x \leq 1)$.

R_2 . The same as R_1 , with v in place of u , and with the rôles of x and y interchanged.

In a previous paper† the author has shown that, under these restrictions, we may express

$$\int_{B(E)} \frac{\partial V_1}{\partial n} ds = \int_{B(E)} u_1 dx + v_1 dy$$

in the form

$$\int_E d\alpha_1.$$

In the present paper it is further proved that

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dx dy.$$

Consider a rectangle r ($a \leq x \leq b, c \leq y \leq d$) contained in J . We have

$$\begin{aligned} \int_{B(r)} u_1 dx &= \int_a^b [u_1(x, c) - u_1(x, d)] dx \\ &= - \int_a^b dx \int_c^d d_y u_1(x, y). \end{aligned}$$

In this $\int_c^d d_y u_1(x, y)$ may be regarded as a Stieltjes integral with y as variable, which, by R_1 , exists for nearly all values of x and is summable in x . Then

$$- \int_a^b dx \int_c^d d_y u_1(x, y) = \int_r d\alpha'(x, y)$$

* De la Vallée Poussin, Cours d'Analyse, 3d ed., vol. 1, § 147.

† P. J. Daniell, this BULLETIN, Nov., 1918.

will be an absolutely additive function of rectangles r . Similarly

$$\begin{aligned}\int_{B(r)} v_1 dy &= \int_c^d [v_1(b, y) - v_1(a, y)] dy \\ &= \int_c^d dy \int_a^b d_x v_1(x, y) \\ &= \int_r d\alpha''(x, y),\end{aligned}$$

$$\int_{B(r)} u_1 dx + v_1 dy = \int_r d\alpha(x, y),$$

where $\alpha = \alpha' + \alpha''$.

For nearly all values of x , $(\partial V_2 / \partial y) = -u$ is uniformly bounded with respect to y , by R_1 . Then, for nearly all values of x , $V_2(x, y)$ is an absolutely continuous function of y and must be of limited variation in y .

By a theorem on integration by parts,*

$$\begin{aligned}- \int_c^d d_y(V_2 u_1) &= - \int_c^d V_2 d_y u_1 - \int_c^d u_1 d_y V_2 \\ &= - \int_c^d V_2 d_y u_1 + \int_c^d u_1 u_2 dy,\end{aligned}$$

for nearly all values of x .

Again since V_2 is "differentiable" at every point of J , it is continuous† and uniformly bounded.

If $\max |V_2| = K$, the total variation of $V_2 u_1$, varying $y(0 \leq y \leq 1)$ will be not greater than

$$K \times \text{variation of } u_1 + \int_0^1 |u_1 u_2| dy.$$

u_1, u_2 are summable with their squares in J , so that $u_1 u_2$ is also summable in J , or $\int_0^1 |u_1 u_2| dy$ is a summable function of x where it exists (nearly everywhere). Then $V_2 u_1$ will also satisfy R_1 , and similarly $V_2 v_1$ will satisfy R_2 .

* P. J. Daniell, *Transactions Amer. Math. Society*, October, 1918.

† De la Vallée Poussin, *ibid.*

Combining the various facts, we then obtain

$$\begin{aligned}\int_{B(r)} V_2 u_1 dx &= - \int_a^b dx \int_c^d d_y(V_2 u_1) \\ &= - \int_a^b dx \int_c^d V_2 d_y u_1 + \int_a^b dx \int_c^d u_1 u_2 dy \\ &= \int_r V_2 d\alpha' + \int_r u_1 u_2 dx dy.\end{aligned}$$

The change from repeated to double integrals is legitimate, in the first integral (Stieltjes) because V_2 is uniformly continuous, in the second because $u_1 u_2$ is summable in J . Similarly

$$\int_{B(r)} V_2 v_1 dy = \int_r V_2 d\alpha'' + \int_r v_1 v_2 dx dy.$$

Then

$$\int_{B(r)} V_2 (u_1 dx + v_1 dy) = \int_r V_2 d\alpha_1 + \int_r (u_1 u_2 + v_1 v_2) dx dy.$$

This is an equation in which all three expressions are absolutely additive functions of rectangles, and $V_2 u_1$, $V_2 v_1$ satisfy R_1 , R_2 respectively; therefore for any set E , measurable Borel, contained in J ,

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dx dy.$$

This was the theorem to be proved and we can rewrite it in the form

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) dS.$$

Corollary 1. Interchanging V_1 , V_2 and subtracting,

$$\int_{B(E)} \left(V_2 \frac{\partial V_1}{\partial n} - V_1 \frac{\partial V_2}{\partial n} \right) ds = \int_E V_2 d\alpha_1 - \int_E V_1 d\alpha_2.$$

Corollary 2. Instead of this make $V_1 = V_2 = V$.

$$\int_{B(E)} V \frac{\partial V}{\partial n} ds = \int_E V d\alpha + \int_E \text{grad}^2 V dS.$$

Three dimensions. The case of three dimensions is more valuable in applications, and the proof is exactly similar. We content ourselves with the mere statement. [Our previous notation is altered; u_1 takes the place of v_1 , and v_1 of $-u_1$.] Let V_1, V_2 be two potential functions, defined and "differentiable" in the cube J ($0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$); let the six partial derivatives be summable with their squares in J ; and let $\partial V_1/\partial x, \partial V_2/\partial x$ satisfy R_1 ; $\partial V_1/\partial y, \partial V_2/\partial y$ satisfy R_2 ; $\partial V_1/\partial z, \partial V_2/\partial z$ satisfy R_3 .

R_1 . The total variation of u varying x ($0 \leq x \leq 1$) is a function of (y, z) finite nearly everywhere and summable in $(0 \leq y \leq 1, 0 \leq z \leq 1)$.

R_2, R_3 . The same as R_1 with v, w in place of u and with cyclical interchanges of x, y, z .

If the element of normal is drawn outwards,

$$\int_{B(r)} \frac{\partial V_1}{\partial n} dS = \int_r d\alpha_1$$

is an absolutely additive function of rectangular parallelepipeds r and $\alpha_1(x, y, z)$ is a function of limited variation in J . Then we can define for any set E , in J , measurable Borel,

$$\int_{B(E)} \frac{\partial V_1}{\partial n} dS = \int_E d\alpha_1$$

and Green's theorem becomes

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} dS = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) d \text{ vol.}$$

The two corresponding corollaries follow immediately.

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ROTATING CYLINDERS AND RECTILINEAR VORTICES.

BY PROFESSOR H. BATEMAN.

§ 1. *Rectilinear Vortex and Rotating Cylinder in a Stream of Incompressible Fluid.*

WE shall assume that the rotation of the cylinder produces a circulation around the cylinder which may be approximately represented by placing a rectilinear vortex along the axis of the cylinder, a device which was adopted by Lord Rayleigh* in his paper "On the irregular flight of a tennis ball." Let V be the velocity of the stream, $2\pi k$ the circulation around the cylinder and $2\pi c$ the strength of the rectilinear vortex at the side of the cylinder. Assuming that the axis of this vortex is parallel to the axis of the cylinder, the motion is two-dimensional and we may represent the velocity potential ϕ and stream line function ψ as follows:

$$\phi + i\psi = U \left(z + \frac{a^2}{z} \right) + ik \log z + ic \log \frac{z - z_0}{z - z_1},$$

where a is the radius of the cylinder, $z = x + iy$ is a complex variable specifying the position of a point relative to the axis of the cylinder, z_0 specifies the position of the vortex, and z_1 that of its image.

Let (u, v) be the component velocities of the fluid at (x, y) , (u_0, v_0) the component velocities of the vortex; then differentiating the above equation with regard to z we find that

$$u - iv = U \left(1 - \frac{a^2}{z^2} \right) + \frac{ik}{z} + ic \left[\frac{1}{z - z_0} - \frac{1}{z - z_1} \right].$$

In calculating (u_0, v_0) we ignore the infinite velocity produced by the vortex itself, consequently

$$u_0 - iv_0 = U \left(1 - \frac{a^2}{z_0^2} \right) + \frac{ik}{z_0} - ic \frac{1}{z_0 - z_1}.$$

* *Mess. of Math.* (1878); Scientific Papers, vol. 1, p. 344. See also Lamb's *Hydrodynamics*, 4th ed., 1916, p. 77; Greenhill, *Mess. of Math.*, vol. 9 (1880), p. 113. Report on Gyroscopic Theory, Report of the Advisory Committee for Aeronautics, No. 146, p. 238.

Let us write

$$z_0 = r_0 e^{i\theta_0}, \quad \zeta_0 = r_0 e^{-i\theta_0}, \quad z_1 = \frac{a^2}{r_0} e^{+i\theta_0};$$

then since

$$\frac{d\zeta_0}{dt} = u_0 - iv_0,$$

we find that

$$\begin{aligned} \frac{dr_0}{dt} &= U \left(1 - \frac{a^2}{r_0^2} \right) \cos \theta_0, \\ r_0 \frac{d\theta_0}{dt} &= -U \left(1 + \frac{a^2}{r_0^2} \right) \sin \theta_0 - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2}. \end{aligned}$$

In order that the vortex may be stationary we must have either $U = 0$ or $\cos \theta_0 = 0$. The former case has been discussed by Greenhill* both for a stationary and moving vortex. In the latter case U , k , c , and r_0 must be connected by the equation

$$(1) \quad \mp U \left(1 + \frac{a^2}{r_0^2} \right) - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2} = 0.$$

To study the stability of this stationary vortex let R , Θ denote small displacements from the stationary position; then

$$\begin{aligned} \frac{dR}{dt} &= \mp U \Theta \left(1 - \frac{a^2}{r_0^2} \right), \\ r_0 \frac{d\Theta}{dt} &= \pm 2UR \frac{a^2}{r_0^3} + \frac{k}{r_0^2} R - cR \frac{r_0^2 + a^2}{(r_0^2 - a^2)^2}, \\ \therefore \frac{d^2 R}{dt^2} &= \mp U \left(1 - \frac{a^2}{r_0^2} \right) \left[\pm 2U \frac{a^2}{r_0^4} + \frac{k}{r_0^3} - \frac{c(r_0^2 + a^2)}{r_0(r_0^2 - a^2)^2} \right] R. \end{aligned}$$

When the upper sign is taken [$\theta_0 = +(\pi/2)$], the vortex is in stable equilibrium if

$$(2) \quad 2U \frac{a^2}{r_0^3} + \frac{k}{r_0^2} > \frac{c(r_0^2 + a^2)}{(r_0^2 - a^2)^2}.$$

The velocity at the surface of the cylinder is determined from the formula

* Encyclopedia Britannica. Article on "Hydrodynamics." The present case may possibly also have been discussed by Greenhill in a paper, *Quart. J. Math.*, vol. 15 (1878), which is at present inaccessible to me.

$$(3) \quad u^2 + v^2 = \left[2U \sin \theta + \frac{k}{a} + \frac{c}{a} \frac{a^2 - r_0^2}{a^2 - 2ar_0 \cos(\theta - \theta_0) + r_0^2} \right]^2.$$

The component forces (X , Y) on the cylinder are given by the formula

$$X - iY = \frac{1}{2}\rho \int_0^{2\pi} (u^2 + v^2) a e^{-i\theta} d\theta + \rho \int_0^{2\pi} \frac{\partial \phi}{\partial t} a e^{-i\theta} d\theta.$$

Expanding $u^2 + v^2$ by Fourier's theorem, we find that the terms involving $\cos \theta$ and $\sin \theta$ are

$$\begin{aligned} 4U \frac{k}{a} \sin \theta - 4U \frac{c}{a} \sin \theta - 4 \frac{kc}{ar_0} \cos(\theta - \theta_0) \\ + 4U \frac{ca}{r_0^2} \sin(\theta - 2\theta_0) + 4 \frac{c^2 r_0 \cos(\theta - \theta_0)}{a(r_0^2 - a^2)}; \end{aligned}$$

hence the value of the first integral is

$$X - iY = -2\pi\rho i [Uk - c(u_0 - iv_0)].$$

On the other hand we have for $r = a$

$$\begin{aligned} \frac{\partial \phi}{\partial t} + i \frac{\partial \psi}{\partial t} &= -ic \frac{u_0 + iv_0}{ae^{i\theta} - r_0 e^{i\theta_0}} + ic \frac{u_1 + iv_1}{ae^{i\theta} - (a^2/r_0)e^{i\theta_0}}, \\ \int_0^{2\pi} \frac{ae^{i\theta} d\theta}{ae^{i\theta} - r_0 e^{i\theta_0}} &= 0, \quad \int_0^{2\pi} \frac{ae^{-i\theta} d\theta}{ae^{i\theta} - r_0 e^{i\theta_0}} = -2\pi \frac{a^2}{r_0^2} e^{-2i\theta_0}, \\ \int_0^{2\pi} \frac{ae^{i\theta} d\theta}{ae^{i\theta} - (a^2/r_0)e^{i\theta_0}} &= 2\pi, \quad \int_0^{2\pi} \frac{ae^{-i\theta} d\theta}{ae^{i\theta} - (a^2/r_0)e^{i\theta_0}} = 0, \\ \int_0^{2\pi} a(\dot{\phi} + i\dot{\psi}) \cos \theta d\theta &= \pi ic \frac{a^2}{r_0^2} (u_0 + iv_0) e^{-2i\theta_0} + \pi ic(u_1 + iv_1), \\ \int_0^{2\pi} a(\dot{\phi} + i\dot{\psi}) \sin \theta d\theta &= -\pi c \frac{a^2}{r_0^2} (u_0 + iv_0) e^{-2i\theta_0} + \pi c(u_1 + iv_1), \\ \int_0^{2\pi} \frac{\partial \phi}{\partial t} a \cos \theta d\theta &= \pi c \frac{a^2}{r_0^2} (u_0 \sin 2\theta_0 - v_0 \cos 2\theta_0) \\ &\quad - \pi c v_1 = -2\pi c v_1, \\ \int_0^{2\pi} \frac{\partial \phi}{\partial t} a \sin \theta d\theta &= -\pi c \frac{a^2}{r_0^2} (u_0 \cos 2\theta_0 + v_0 \sin 2\theta_0) \\ &\quad + \pi c u_1 = 2\pi c u_1. \end{aligned}$$

Hence we finally obtain the formulas

$$X = 2\pi\rho c(v_0 - v_1), \quad Y = 2\pi\rho(Uk - cu_0 + cu_1).$$

For a stationary vortex $X = 0$, $Y = 2\pi\rho Uk$ and the transverse force is the same as in Rayleigh's case when $c = 0$ and there is no vortex.*

For a stationary vortex (1) and (2) give the inequality

$$-U > \frac{2ca^2r_0^3}{(r_0^2 - a^2)^3};$$

hence if U is positive and $\theta_0 = \pi/2$, c must be negative and k negative. This means that the circulations around the vortex and cylinder are both in the counterclockwise direction and that the force $2\pi\rho Uk$ is negative. Hence this force tends to move the cylinder away from the region where the cylinder and stream of air are moving in opposite directions. This phenomenon attracted the attention of Newton in 1671 and was the subject of some experiments by Magnus in 1852. Many other experiments have been made since this time and in his beautiful lecture on† "The dynamics of a golf ball" Sir Joseph Thomson showed with the aid of a pressure gauge that there is indeed a difference of pressure on the two sides of a golf ball rotating in a stream of air and remarked that when a golf ball is in flight this difference in pressure may provide a lifting force greater than the weight of the ball, so that a rotating golf ball is a kind of flying machine. This

* In this case the formula gives Rayleigh's law that the transverse force is proportional to the velocity of the stream relative to the cylinder and the velocity of spin. In the case of a ball this product must be multiplied by the sine of the angle between the direction of motion of the ball relative to the air and the axis of spin. Rayleigh's law was adopted by P. G. Tait (*Nature*, June 29, 1893; *Papers*, vol. 2, p. 386) in his mathematical calculation of the trajectories of a golf ball and by Sir J. J. Thomson in his experimental method (*Nature*, Dec. 22, 1910) of imitating these trajectories by means of the path of an electron in superposed electric and magnetic fields. A slightly different law is adopted by Appell (*Journ. de Physique*, vol. 7 (1917), p. 5) in his analysis of Carrière's experiments, *ibid.*, vol. 5 (1916), p. 175. The formula has been established for a cylindrical surface of arbitrary shape in a stream of fluid on the assumption that there is a circulation round the cylinder. This is the basis of the theory of sustentation developed by Kutta, Joukowski and others (see Joukowski, *Aérodynamique*, Paris, 1916, and a paper by R. Jones, *Proc. Roy. Soc. London*, vol. 92, A (1916), p. 107) and of the theory of propeller action which has been developed by R. Grammel (*Jahrb. d. Schiffsbau-technik*, vol. 17 (1916), p. 367).

† Royal Institution, March 18 (1910); *Nature*, Dec. 22 (1910).

effect of rotation has been used to increase the range of spherical projectiles and in his experiments with golf balls* Tait conclusively proved that the great factor in long driving was the underspin communicated to the ball by the impact of the club.

Lafay† has recently determined the way in which the pressure varies over the surface of a cylinder rotating rapidly in a stream of air and finds that when the transverse force is in the direction indicated by Magnus it is produced chiefly by the suction on the side which is moving in the same direction as the stream of air.‡ The analogy with the wing of an aeroplane is thus complete.

The distribution of pressure is important, for it should indicate whether or not a rotating cylinder carries along with it one or more vortices that do not produce any circulation around the cylinder.

Lafay's results for a speed of rotation of 9,450 revolutions per minute and a velocity of the air stream of 19 meters per second are given below (p) and compared with the corresponding results for the case of no rotation (p_0). The pressures are in millimeters of water and represent deviations from the atmospheric pressure.

θ	0	15	30	45	60	75	90
p	-18	-16	-12	-9	-8	-5	-2
p_0	-10	-10	-10	-10	-11	-13	-17
		105	120	135	150	165	180
		3	8	14	20	22	18
		-21	-21	-13	7	20	23
θ	-0	-15	-30	-45	-60	-75	-90
p	-18	-23	-32	-63	-93	-102	-104
p_0	-10	-10	-10	-11	-12	-14	-17
		-105	-120	-135	-150	-165	-180
		-85	-63	-37	-14	7	18
		-20	-21	-13	7	19	23

* *Badminton Magazine*, March (1896).

† *Comptes Rendus*, vol. 153 (1911), p. 1472.

‡ The experiments and remarks made by W. S. Franklin, *Journ. Franklin Inst.*, vol. 177 (1914), p. 23, are of some interest in this connection.

To compare these results with formula (3), we must take into account the fact that the theory gives no drag X when $k = 0$ and $c = 0$. It seems reasonable to suppose, however, by analogy with (3) that the correct formula is of type

$$u^2 + v^2 = \left[f(\theta) + \frac{k}{a} + \frac{c}{a} \frac{a^2 - r_0^2}{a^2 - 2ar_0 \cos(\theta - \theta_0) + r_0^2} \right]^2,$$

where $f(\theta)$ is some function which gives the distribution of velocity in the case when there is no rotation of the cylinder. Now if c were zero the above formula would indicate that $\sqrt{P - p}$ should differ from $\sqrt{P - p_0}$ by a constant proportional to k/a , where P is some constant. This means that

$$(p - p_0)^2 + \lambda(p + p_0)$$

should be constant where λ is some constant. This, however, is far from the case as may be seen from some of the values of $(p - p_0)^2$ and $p + p_0$

$(p - p_0)^2$	64	36	4	1	9	8100	441	25
$p + p_0$	-28	-26	-22	-21	-29	-118	-7	41

It seems reasonable then to assume that the flow is modified by the presence of one or more vortices and if we wish to try to account for the drag on the cylinder with the aid of these vortices it is necessary to assume that they are in motion relative to the cylinder just as in Kármán's theory of resistance.

It should be noticed that the region of low pressure in Lafay's experiment occurs in the neighborhood of $\theta = -90^\circ$, and so is directly opposite to the region where a vortex can remain in stable equilibrium. Hence if a vortex forms in the region of low pressure* it cannot remain stationary.

If c is negative and u_0 positive so that the vortex is carried away by the stream, Y may be decreased in magnitude and may even be positive instead of negative. A vortex with counterclockwise rotation may perhaps form when U is greater than the circumferential velocity of the rotating cylinder. As before, we suppose that the rotation of the

* Lord Kelvin pointed out that if the velocity of a spherical solid moving through a fluid exceeds a certain value the pressure becomes negative when calculated by the ordinary theory and so cavitation must commence at the back of the sphere; coreless vortices will be formed periodically and shed off behind the sphere during its motion through the fluid. *Phil. Mag.*, vol. 23 (1887).

cylinder is counterclockwise. If on the other hand c is positive and u_0 positive our formula shows that Y is negative and numerically greater than in the case when there is no vortex. A vortex with clockwise rotation may, perhaps, form in the neighborhood of $\theta = -90^\circ$ when the circumferential velocity of the rotating cylinder is greater than the velocity of the stream.

To account for the force on the cylinder in the direction of the axis of x it is necessary to suppose that a vortex with counterclockwise rotation moves away from this axis and that a vortex with clockwise rotation moves towards this axis.

It should be mentioned that Lafay has found* that the direction of the transverse force in the Magnus experiment could be reversed. Experimenting with a smooth aluminum cylinder 35 cm. long and 10 cm. in diameter, he found that if the velocity of the air stream were kept constant at 18 or 19 meters per second and the velocity of rotation gradually increased, the direction of the force on the cylinder first swung to one side of the air stream, attained a maximum inclination to it of about 11° , then swung to the other side, attained a maximum inclination of about 57° , and finally appeared to approach asymptotically to a direction making an angle of 45° with the air stream.

The maximum inverse effect occurred when the cylinder was rotating at a speed of 1570 turns per minute giving a circumferential velocity of about 8.22 meters per second which is less than the velocity of the air stream. On the other hand the direct effect was quite marked when the speed of rotation was 9450 turns per minute, in which case the circumferential velocity is in the neighborhood of 50 meters per second and is greater than the velocity of the air stream. This is exactly in accordance with the above view that the transverse force is modified by the production of vortices in the neighborhood of the region $\theta = -90^\circ$ and that the direction of rotation in the vortices depends upon whether the circumferential velocity is greater or less than that of the stream of air.

It is well known that the drift of a projectile fired from a rifled gun is exactly opposite to the direction of the transverse force which is indicated by the normal Magnus effect, but here we are dealing with a case in which the component velocity of translation of the bullet in a direction perpendicular to its axis is perhaps at some time greater than the circumferential

* *Comptes Rendus*, vol. 151 (1910), p. 867; vol. 153 (1911), p. 1472.

velocity due to its spin and so the transverse force may be reversed as in Lafay's experiments. In Kármán's theory of resistance* it is supposed that vortices with opposite senses of rotation are formed alternately behind a cylinder and move down the stream in two rows at a certain distance apart. These vortices occupy the region of the "wake" behind a cylinder in a stream of fluid. Now in Lanchester's theory of the Magnus effect† it is assumed that the wake behind the cylinder is displaced to one side on account of the rotation of the cylinder. If the vortices in this wake are produced somewhere in the neighborhood of $\theta = -90^\circ$, the displacement of the wake to one side would be accounted for by the previous remark that a vortex with counterclockwise rotation moves away from the axis of x and that a vortex with clockwise rotation moves towards‡ the axis of x while it is carried down the stream.§ It should be mentioned, however, that here we imagine the resistance of the cylinder to arise from the velocities of the vortices in a direction at right angles to the stream, while in Kármán's theory the resistance arises from the momentum which is carried away from the cylinder whenever a fresh pair of vortices is formed. This momentum is calculated from the circulations around the vortices and the distance between the two rows, while the rate at which the vortices are formed is calculated from their final distance apart in a row and their final velocity relative to the cylinder which is now parallel to the axis of x . The difference between the two points of view is probably the same as the difference between the initial and final stages of an action; for, when the motion of the vortex perpendicular to the stream is considered, we are dealing with the actual transfer of momentum from the cylinder to the fluid or vice-versa.

§ 2. *Two Rectilinear Vortices and a Rotating Circular Cylinder in a Stream of Fluid.*

An interesting attempt to throw light on the initial stages of the formation of the two rows of vortices in the Kelvin-

* *Phys. Zeitschr.* (1912), *Gött. Nachr.* (1911). See also Joukowsky *Aérodynamique*, Paris (1916), Ch. 8; Lamb, *Hydrodynamics*, 4th ed. (1916), p. 219.

† *Aerodynamics*, vol. I.

‡ When vortices originate on opposite sides of the x -axis as in the case of no rotation they must both move away from the axis of x if they are to produce a positive x .

§ If it crosses the axis of x it must then move away from it.

Kármán theory of resistance has been made by L. Föppl* who has shown that when a cylinder is at rest in a stream of fluid two vortices which are images of one another in the axis of x can be in equilibrium in a stationary position behind the cylinder provided they lie on the curves

$$\pm 2r^2 \sin^2 \theta = r^2 - a^2.$$

Föppl found that the vortices are stable for symmetrical displacements but unstable for asymmetric displacements. The former result is of some meteorological interest in connection with the flow of air past a mountain or other obstacle which is shaped roughly like a half cylinder standing on a plane. It is well known in fact that an eddy can form behind a mountain over which a wind is blowing†; near the rock of Gibraltar the eddy motion is sometimes quite large.

The second result is of interest because it indicates that if two vortices form in symmetrical positions behind a cylinder as they do behind a flat plate, they will be in unstable equilibrium for asymmetric displacements; consequently one vortex may be imagined to get ahead of the other and a new one to be formed to take its place, thus giving rise to an alternate formation as imagined by Lord Kelvin and Kármán.‡ It should be mentioned that in Föppl's analysis the strength of the vortices when in equilibrium increases with their distance from the cylinder while in Kármán's analysis the strengths of all the vortices in one row are supposed to be the same and equal but opposite in sign to those of the vortices in the other parallel row.

If $z_0 = r_0 e^{i\theta_0}$ indicates the position of one of the vortices and c the strength, we have in Föppl's case

$$c = U \left(1 - \frac{a^2}{r_0^2} \right)^2 \left(r_0 + \frac{a^2}{r_0} \right)$$

* *München Sitzungsberichte* (1913).

† See, for instance, W. H. Dines, Report of the Advisory Committee for Aeronautics, No. 92, March (1913), W. N. Shaw, *Science Progress*, vol. 6, p. 345.

‡ This alternate formation of two rows of vortices has been observed on many occasions. See for instance Osborne Reynolds, *Phil. Trans.*, vol. 174 (1883); Ahlborn, "Ueber den Mechanismus des hydrodynamischen Widerstandes," Hamburg (1902); Mallock, *Proc. Roy. Soc.*, vol. 79 (1907), p. 262; vol. 84 (1910), p. 490; *Engineering*, April 19 (1912); H. Bénard, *Comptes Rendus*, vol. 147 (1908), pp. 839, 970; vol. 156 (1913), p. 1003; vol. 157, pp. 7, 89, 171; Rayleigh, *Phil. Mag.*, vol. 29 (1915), p. 433. Joukowski, loc. cit.

and the velocity at the surface of the cylinder is given by the formula

$$u^2 + v^2 = \left[2U \sin \theta - \frac{4cr_0(r_0^2 - a^2) \sin \theta \sin \theta_0}{\{r_0^2 + a^2 - 2ar_0 \cos(\theta - \theta_0)\} \{r_0^2 + a^2 - 2ar_0 \cos(\theta + \theta_0)\}} \right]^2.$$

Since $2r_0^2 \sin \theta_0 = r_0^2 + a^2$ it appears that the velocity vanishes when $\theta = 0$, $\theta = \pi$ and also when

$$\cos \theta = \frac{1}{2a} \left(r_0 + \frac{a^2}{r_0} \right) \cos \theta_0 - \frac{1}{a} \sqrt{3r_0^2 + 4a^2} \sin^2 \theta_0.$$

This equation can give a real value of θ , for when $r_0^2 = 3a^2$, we have

$$\cos \theta = \frac{4\sqrt{6} - \sqrt{13}}{9} < 1.$$

When the surface velocity q is plotted as a function of θ , q and θ being regarded as polar coordinates, a curve is obtained which is shaped like a butterfly with two wings. The measurements of J. T. Morris,* A. Thurston† and A. Lafay‡ indicate that for a cylinder in a stream of fluid the curve indicating the distribution of velocity should be shaped like a butterfly with only one pair of wings. The lack of agreement is to be expected on account of the instability of the two vortices behind the cylinder. A comparison of the theoretical formula ought to be made with some measurements of the velocity over the surface of a semi-cylinder standing on a smooth plane surface over which a wind is blowing. Of course the roughness of a rotating ball or cylinder has a great influence on the magnitude of the transverse force exerted by the wind as is clearly shown in the experiments of Sir Ralph Payne-Gallwey,§ Sir Joseph Thomson and Commandant Lafay. Curiously enough the former found that to obtain the best lifting effect with a golf ball the ball must not be too rough. This suggests that with a very rough ball vortices are produced which modify the Magnus effect.

* *Engineering*, Aug. 8, 1913.

† *Ibid.*, Aug. 21, 1914.

‡ *Loc. cit.*

§ *The Times*, March 16, 23, 1909. *Nature*, April 22, 1909.

It may be of interest to indicate briefly the extension of Föppl's analysis for the case in which the cylinder rotates and there is no symmetry about the axis of x . Using z_0 and z_2 to indicate the positions of the vortices and z_1 and z_3 those of their images in the cylinder, the appropriate expressions for ϕ and ψ are given by

$$\phi + i\psi = U \left(z + \frac{a^2}{z} \right) + ik \log z + ic \log \frac{z - z_0}{z - z_1} - ic \log \frac{z - z_2}{z - z_3}.$$

Differentiating to obtain the velocities we have

$$u - iv = U \left(1 - \frac{a^2}{z^2} \right) + \frac{ik}{z} + ic \left[\frac{1}{z - z_0} - \frac{1}{z - z_1} \right] - ic \left[\frac{1}{z - z_2} - \frac{1}{z - z_3} \right],$$

$$u_0 - iv_0 = U \left(1 - \frac{a^2}{z_0^2} \right) + \frac{ik}{z_0} - ic \left[\frac{1}{z_0 - z_1} + \frac{1}{z_0 - z_2} - \frac{1}{z_0 - z_3} \right],$$

$$u_2 - iv_2 = U \left(1 - \frac{a^2}{z_2^2} \right) + \frac{ik}{z_2} + ic \left[\frac{1}{z_2 - z_0} - \frac{1}{z_2 - z_1} + \frac{1}{z_2 - z_3} \right].$$

Writing

$$z_0 = r_0 e^{i\theta_0}, \quad z_2 = r_2 e^{i\theta_2}, \quad R^2 = r_0^2 + r_2^2 - 2r_0 r_2 \cos(\theta_0 - \theta_2),$$

$$S^2 = r_0^2 r_2^2 + a^4 - 2a^2 r_0 r_2 \cos(\theta_0 - \theta_2)$$

we find that when $u_0 = v_0 = u_2 = v_2 = 0$,

$$0 = U \left(1 - \frac{a^2}{r_0^2} \cos 2\theta_0 \right) + \left(\frac{k}{r_0} - \frac{cr_0}{r_0^2 - a^2} \right) \sin \theta_0 + \frac{c}{R^2} (r_2 \sin \theta_2 - r_0 \sin \theta_0) + \frac{cr_2}{S^2} (r_0 r_2 \sin \theta_0 - a^2 \sin \theta_2),$$

$$0 = U \frac{a^2}{r_0^2} \sin 2\theta_0 + \left(\frac{k}{r_0} - \frac{cr_0}{r_0^2 - a^2} \right) \cos \theta_0$$

$$+ \frac{c}{R^2} (r_2 \cos \theta_2 - r_0 \cos \theta_0) + \frac{cr_2}{S^2} (r_0 r_2 \cos \theta_0 - a^2 \cos \theta_2)$$

and two similar equations with the suffixes 0 and 2 interchanged and $-c$ written in place of c .

Multiplying the first of these equations by $\cos \theta_0$, the second by $\sin \theta_0$ and subtracting, we get

$$(r_0^2 - a^2) \left[\frac{U}{r_0^2} \cos \theta_0 - \frac{cr_2}{R^2 S^2} (r_2^2 - a^2) \sin (\theta_0 - \theta_2) \right] = 0.$$

Similarly

$$(r_2^2 - a^2) \left[\frac{U}{r_2^2} \cos \theta_2 - \frac{cr_0}{R^2 S^2} (r_0^2 - a^2) \sin (\theta_0 - \theta_2) \right] = 0,$$

$$\left[r_0 - \frac{a^2}{r_0} \right] \cos \theta_0 = \left(r_2 - \frac{a^2}{r_2} \right) \cos \theta_2.$$

This equation tells us that the projection on the axis of x of the interval between a vortex and its image in the cylinder is the same in both cases. It is clear then that the vortex which is furthest from the plane $y = 0$ is also furthest from the axis of the cylinder. There is another equation connecting r_0 , θ_0 , r_2 , θ_2 , but it is very complicated; the two equations show, however, that when one vortex is given the position of the other is determined. The component forces on the cylinder are easily found by an extension of the analysis of § 1. They are

$$X = 2\pi\rho c(v_0 - v_1 - v_2 + v_3),$$

$$Y = 2\pi\rho(Uk - cu_0 + cu_1 + cu_2 - cu_3).$$

When both vortices are stationary we have $X = 0$, $Y = 2\pi\rho Uk$ as before. This result seems to be true for any number of stationary vortices outside a cylinder, as is probably well known.*

We have seen that a rotation of the cylinder alters the possible stationary positions for a pair of vortices; it may also alter the period of formation of the vortices in Kármán's theory of resistance. This is not easy to settle mathematically but the matter may perhaps be tested experimentally.

Eiffel found during his measurements of the force exerted by a stream of air on a sphere that at a speed above twenty

* See for instance the remark in § 3 of Föppl's paper.

miles an hour the flow was smoother and the force more constant. The vortices behind the sphere are usually said to be flattened out* and presumably the period of formation which depends on the distance between the two rows of vortices is changed. The rate of formation probably increases with the velocity of the stream until the flow becomes practically steady at high speeds.†

Now a rotation of the cylinder or sphere may cause a change in the critical velocity above which the flow is practically steady, consequently it may be worth while to determine this critical velocity for a given velocity of the stream and different velocities of rotation of the cylinder. This critical velocity may be closely connected with Lafay's critical velocity at which the sign of the transverse force in the Magnus effect is reversed; it should be noticed, however, that Lafay made his experiments in a wind of 19 meters a second which had a velocity more than double Eiffel's critical velocity and noticed at which *speed of rotation* the change occurred.

It is known that at high speeds the periodic formation of vortices is responsible for the production of sound‡ and the period is presumably that of the sound. In the case of the aeolian harp§ the sound is most intense when the period is close to one of the natural periods of the stretched string or wire. An effect of rotation of the stretched string or wire on the pitch of the sound produced in a given type of wind might perhaps be determined experimentally but there would be difficulties. For the mathematical theory of the aeolian harp the sound produced by the oscillation of a vortex about a state of uniform motion ought also to be considered, for this may contribute to the observed sound as well as periodic formation of vortices, but the effect is probably negligible.

§ 3. *Vortices in a Compressible Fluid.*

It was shown by Lord Kelvin that vortex lines in a compressible fluid move with the fluid provided the density of

* Cf. Loening, *Military Aeroplanes*, p. 51.

† Cf. Cowley and Levy, *Aeronautics in Theory-Experiment* (1918), pp. 17-20.

‡ See for instance Mallock, *Proc. Roy. Soc. London*, vol. 84 (1910), p. 490.

§ Lord Rayleigh, *Theory of Sound*, vol. II, p. 412, vol. I, p. 212; *Phil. Mag.*, vol. 29 (1915), p. 433.

the fluid is a function of the pressure only and the body forces have a single-valued potential. It is also true that the circulation in any circuit moving with the fluid remains constant. An accurate theory of vortex motion in a compressible fluid is difficult. To make progress, it seems worth while to make an assumption which is nearly true except in the immediate neighborhood of a vortex.

Let us consider a two-dimensional irrotational motion in which the velocity potential ϕ satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2},$$

wherein c , the velocity of sound in the fluid, is supposed to be constant. The component velocities (u , v) will then satisfy the same equation. A solution appropriate for the representation of a rectilinear vortex moving with constant component velocities a , b may be derived from the well known solution for a stationary vortex by an application of the transformations of the theory of relativity. The result is

$$u = - \frac{\mu c (y - bt) \sqrt{c^2 - a^2 - b^2}}{(c^2 - b^2)(x - at)^2 + (c^2 - a^2)(y - bt)^2 + 2ab(x - at)(y - bt)},$$

$$v = \frac{\mu c (x - at) \sqrt{c^2 - a^2 - b^2}}{(c^2 - b^2)(x - at)^2 + (c^2 - a^2)(y - bt)^2 + 2ab(x - at)(y - bt)},$$

where $2\pi\mu$ is the strength of the vortex.*

To find the paths of the particles of fluid relative to the moving vortex, we write $X = x - at$, $Y = y - bt$, $X^2 + Y^2 = z$, $bX - aY = w$, $u = dx/dt$, $v = dy/dt$; then it is easy to see that

$$\frac{1}{2} \frac{dz}{dt} + aX + bY = 0, \quad \frac{dw}{dt} + \mu \frac{c \sqrt{c^2 - a^2 - b^2} (aX + bY)}{c^2 z - w^2} = 0,$$

$$\therefore \frac{dz}{dw} = \frac{2(c^2 z - w^2)}{\mu c \sqrt{c^2 - a^2 - b^2}},$$

$$\therefore z = A e^{2cw/\mu \sqrt{c^2 - a^2 - b^2}}$$

$$+ \frac{1}{c^2} \left[w^2 + \mu \frac{w}{c} \sqrt{c^2 - a^2 - b^2} + \frac{\mu^2}{2c^2} (a^2 - c^2 - b^2) \right],$$

where A is an arbitrary constant.

* μ is positive for counterclockwise rotation.

Thus relative to the moving vortex a particle of fluid appears to describe a curve whose equation is

$$X^2 + Y^2 = A e^{2\alpha(bX - aY)/\mu\sqrt{c^2 - a^2 - b^2}} + \frac{1}{c^2} \left[(bX - aY)^2 + \frac{\mu}{c} \sqrt{c^2 - a^2 - b^2} (bX - aY) + \frac{\mu^2}{2c^2} (c^2 - a^2 - b^2) \right].$$

When $A = -(\mu^2/c^2)(c^2 - a^2 - b^2)$, we obtain an equation which is satisfied by $X = 0$, $Y = 0$ and this represents a curve having an isolated point at the origin which is in accordance with the theorem that the vortex moves with the fluid. As A varies, we obtain first a number of ovals surrounding the origin, then a curve which touches itself on the axis of y ,* after completing the circuit, and then goes to infinity. Finally we obtain a series of curves which lie outside the last one and like it go to infinity. Each curve is described by a point which starts moving almost in the direction of the x -axis, then swings around the origin in the clockwise direction and finally returns to a state of motion very nearly parallel to the x -axis.

Let us now consider two rectilinear vortices of strengths $2\pi\mu$ and $-2\pi\mu$ respectively moving parallel to the axis of x with constant velocity a and at a distance apart equal to $2y$. If the velocity of each vortex is that produced by the other, we have simply

$$a = \frac{\mu c}{2y \sqrt{c^2 - a^2}}$$

or

$$a^2 = \frac{1}{2} c^2 \pm \sqrt{\frac{1}{4} c^2 - \frac{\mu^2 c^2}{4y^2}}.$$

For a real value of a^2 we must have $c^2 y^2 > \mu^2$. Hence when the circulations around the vortices are given they cannot move parallel to one another at a distance apart less than $2\mu/c$. If the two vortices are in a stream of fluid moving with velocity U the equation determining a is

$$a = U + \frac{\mu c}{2y \sqrt{c^2 - a^2}}$$

* We put $y = bX - aY$, $x = aX + bY$ to obtain a curve which is symmetrical with regard to the axis of y .

or

$$4y^2(c^2 - a^2)(a - U)^2 = \mu^2 c^2.$$

It should be noticed that a is numerically less than c whatever the value of U .

The velocity potential of a vortex moving with constant velocity differs by a constant from the function

$$\mu \tan^{-1} \frac{(ay - bx)\gamma}{a(x - at) + b(y - bt)} \equiv \mu \int_{-\infty}^t \frac{ds}{\sqrt{t - s}} B(x, y, s),$$

where

$$\gamma = \left(1 - \frac{a^2 + b^2}{c^2}\right)^{1/2}$$

and

$$A(x, y, s) + iB(x, y, s)$$

$$= \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)} \right]^{1/2},$$

$$A(x, y, s) + iB(x, y, s)$$

$$= \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)} \right]^{1/2}.$$

For a vortex which is moving with component velocities $\xi(\tau)$, $\eta(\tau)$ at time τ and is at an infinite distance from the origin at time $\tau = -\infty$, the natural generalization of the function $B(x, y, s)$ is obtained as follows:

Let τ be defined in terms of x, y, s by the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 = c^2(s - \tau)^2 \quad \tau \leq s,$$

where $\xi'^2(\tau) + \eta'^2(\tau) < c^2$ and primes denote differentiations with respect to τ . Let

$$l(\tau) = \xi'(\tau) + i\sigma\eta'(\tau), \quad c^2 p(\tau) = \xi'^2 + \eta'^2,$$

$$m(\tau) = \eta'(\tau) - i\sigma\xi'(\tau), \quad c^2 \sigma^2 = c^2 - \xi'^2 - \eta'^2;$$

then

$$A(x, y, s) + iB(x, y, s)$$

$$= \left[\frac{c^2 p(\tau)}{\{x - \xi(\tau)\}l(\tau) + \{y - \eta(\tau)\}m(\tau) - c^2(s - \tau)p(\tau)} \right]^{1/2}.$$

This result may be of interest for an analysis of the sound produced when a vortex oscillates about a state of uniform motion. It must be remembered, however, that the above analysis is only approximate, for velocities are treated as small in the derivation of the wave equation.

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SHORTER NOTICES.

Lectures on the Philosophy of Mathematics. By JAMES BYRNIE SHAW. Chicago, The Open Court Publishing Company, 1918. viii + 206 pp.

THE purpose of Professor Shaw's book is a discussion of the evergreen question: What is mathematics? While in his first chapter the author develops in a highly exalted style various aspects of this subject, the greater part of the subsequent chapters may be said to be essentially devoted to two more specific questions, viz., what influences operate and have operated in the development of mathematics, and how may existing mathematics be concisely described. With the treatment of these questions, perhaps not always recognized as explicit and distinct, Chapters II to XIII are taken up, together occupying 140 pages. To the first question the author gives a positive answer, viz.: "Mathematics is a creation of the mind and is not due to the generalization of experiences or to their analysis; nor is it due to an innate form or mold which the mind compels experience to assume, but is the outcome of an evolution, the determining factors of which are the creative ability of the mind and the environment in which it finds the problems which it has to solve in some manner and to some degree." The second question is answered in a negative sense; as the various fields and principles of mathematics are discussed, the conclusions are reached that mathematics is not wholly arithmetic, nor geometry, nor logistic; that mathematics can not be completely characterized as a theory of invariance, nor as a theory of functions, etc., however important each of these principles may be. The closest approach to a satisfactory answer to our second

question the author appears to find in the statement that mathematics is a theory of equations. On page 152 we read: "This is the most important central principle of mathematics, namely, that of inversion, or of creating a class of objects that will satisfy certain defining statements. If the mathematician does not find these at hand in natural phenomena, he creates them and goes on in his uninterrupted progress. This might be considered to be *the* central principle of mathematics, for with the new creation we start a new line of mathematics, just as the imaginary started the division of hyper-numbers, just as the creation of the algebraic fields started a new growth in the theory of numbers."

Chapters XIV, XV and XVI take up respectively "Sources of mathematical reality" (15 pages), "The methods of mathematics" (17 pages) and "Validity of mathematics" (10 pages). There is finally given on pages 196 and 197 a double entry table for the classification of existing mathematics, presenting in schematic form the division of the whole field on the basis of which the discussion is carried on in Chapters II to XIII. This very interesting and suggestive mode of dividing mathematics is not the least valuable part of the book. Each mathematical topic is classified as to its structure and as to its central principle. Structurally, mathematics falls into two main divisions, static mathematics and dynamic mathematics. In the former group we find arithmetic, geometry, tactic and logistic; in the latter group are placed algorithms, algebra, transmutations and inferences. The central principles of mathematics are taken to be form, invariance, functionality and inversion.

The book is written in an easy, flowing style; sometimes, as for instance in the first six pages in a flowering style, reminiscent of an earlier literary epoch. In places it is sketchy, so much so as to be of little value to the non-professional reader for whom the book is intended, and to be somewhat irritating to the mathematician. A rather strong example of this is found on page 26, where we find the following statement: "By introducing a 'measure of a set' Lebesgue and others have found a means of handling sets satisfactorily"; others are to be found on pages 90, 91, and at the end of chapter VII. This floweriness and sketchiness of style have in a few instances led to what seems to the reviewer to be an overstatement of perfectly sound positions. As such we

would characterize the assertion that "Nothing whatever in our sense-data tells us that the earth is rotating" (page 156), and the claim that "we find in mathematics that subject whose results have lasted through the vicissitudes of time and are regarded universally as the most satisfactory truths the human race knows" (page 154). Why is Lie relegated to the remote period of the Norsemen?

While the author "cherishes the hope that the professional philosopher too may find some interest in these lectures," he addresses himself primarily to "students of fair mathematical knowledge," such as may be secured through an ordinary college course in mathematics. The book is intended to be non-technical from the philosophical as well as from the mathematical point of view. Indeed it seems to the reviewer that the author has taken the term "philosophy of mathematics" in the more or less popular sense, in which it means general discussion about the nature of mathematics, rather than in the sense of systematic, rigorous, scientific discussion. This being assumed, and the assumption is not intended as a criticism of the value of the book, it is understood why one finds nowhere a statement of the author's general philosophical position, of his beliefs and convictions on some of the fundamental questions of philosophy, on which many of his conclusions are based, and a knowledge of which is indispensable for an understanding and just appraisal of his work. Hints are found scattered throughout the book, as for instance on pages 30, 55, 59, 88, etc.; and perhaps the most definite statement of a philosophical credo is that on page 166: "The mind, it is true, as Kant insisted, organizes experience, but it does this by methods that are evolutionary. It originates schemes from its own activity, and makes a choice of which of several equally valid schemes it will use." But it seems clear that a systematic, critical examination of the philosophical position and of the conclusions derived therefrom would be entirely out of keeping with the purpose of this book.

While granting without hesitation, particularly at the present stage of development of the subject, the value of a general, let us say intuitional, discussion of the philosophy of mathematics, the reviewer would like to use this opportunity to call attention to the scientific aspects of this subject. To these may be said to belong (a) the study of the logical structure of mathematics; (b) the philosophical discussion of concepts to

which mathematics gives rise, such as infinity, countability; (c) the introduction of a mathematical point of view, and as far as possible, of mathematical methods into the discussion of problems belonging to the domains of epistemology, of ethics, and to other philosophical fields. For progress in the treatment of these questions the closest cooperation between philosophers and mathematicians is essential. Such cooperation will certainly be stimulated by the reading of Professor Shaw's book.

ARNOLD DRESDEN.

Functions of a Complex Variable. By THOMAS M. MACROBERT.
London and New York, Macmillan, 1917. xiv + 298 pp.

THIS book is designed for students who have acquired a good working knowledge of the calculus and desire to become acquainted with the theory of functions of a complex variable and its principal applications. The material has been well chosen for accomplishing this purpose. The first two chapters are intended to familiarize the student with the geometrical representation of complex numbers in a plane and with simple rational and irrational functions of a complex variable. Some fundamental properties of holomorphic functions are established in Chapter III and these are used in defining certain elementary transcendental functions. Chapters IV to VII deal in order with the theory and practice of integration, convergence of series and the Taylor and Laurent expansions, uniformly convergent series and infinite products, and various summations and expansions. Up to this point (page 131) the book gives to the student an introduction to the elementary parts of the classic theory of functions of a complex variable with some of the simpler applications.

The theory of the gamma functions is presented in Chapter VIII (pages 132-159) in a way which is not very satisfying. The development of the initial elementary properties of elliptic functions is to be found in Chapters IX, X, XI (pages 160-207) in an exposition which is pleasing and particularly well suited to the needs of the young learner. The remaining chapters (IX to XV, pages 208-276) are devoted to differential equations, in part to general theorems and in other part to special cases, such as the equations of Legendre and Bessel. This division of the book also is well suited to the needs of the beginner.

Besides numerous worked examples the book contains many

problems. In addition to those interspersed throughout the text, there are to be found at the ends of the chapters and in a set of miscellaneous examples at the end of the book no fewer than 550 problems. Many of these are quite elementary in character and serve merely to familiarize the reader with the notions involved in the text. Many others contain significant theorems which would be demonstrated in a more extensive treatise.

Concerning the character of his exposition the author says in his preface: "In order to avoid making the subject too difficult for beginners, I have abstained from the use of strictly arithmetical methods, and have, while endeavoring to make the proofs sufficiently rigorous, based them mainly on geometrical conceptions." It is evident that the desirable golden mean here is a matter on which there is likely to be difference of opinion. The reviewer believes that the statements are sometimes too loose. Thus "closed region" is first defined on page 92. Yet several times on the earlier pages statements have been made involving the word "region" which are correct only when one understands "closed region." For cases in point see page 24, line 6 up; page 26, line 7; page 39, line 7 up. In theorem 2 of page 93 it should be specified that the path of integration is finite in length or the proof should be modified. For theorem 2 on page 39 one should read "No [closed] region can contain an infinite number of isolated singularities [and no other singularities]," the words in brackets being those which one must insert into the theorem to make it accurate. These may be taken as examples of loose statements which are all too frequent, especially in the first 131 pages, in which the more general matters are treated.

A student who has been forewarned against these somewhat loose statements will find the book one by the reading of which he will be much profited. Many persons will probably find it also a suitable source of elegant elementary problems for enforcing a clear understanding of numerous fundamental notions and theorems.

R. D. CARMICHAEL.

Leçons sur les Fonctions Elliptiques en Vue de leurs Applications. Par R. DE MONTESSUS DE BALLORE. Paris, Gauthier-Villars, 1917.

THE object of these lectures, which M. de Montessus de Ballore delivered at the Faculté des Sciences of Paris in 1915—

1916, is to present in an elementary manner the fundamental properties of the ordinary elliptic functions. The first part deals with the Jacobian functions sn , cn , dn , and the elliptic integrals, the second part with the Weierstrassian forms $\wp u$, ζu , σu , and the third part with the general properties of elliptic functions and their applications to the Jacobian and Weierstrassian forms.

In the fourth and last part, the θ -functions are considered.

As may be expected from the announcement in the preface, the book does not contain anything beyond well known elementary propositions. In view of possible applications, of which the book does not contain any, particular attention is given to the construction of formulas for numerical computations.

The reviewer must confess that he would have expected a course of lectures on elliptic functions at the University Paris planned from a higher point of view, as is customary at that famous center of mathematical activity. For this reason the *Leçons*, in spite of their excellence of execution and typography, are disappointing.

ARNOLD EMCH.

NOTES.

THE April number (volume 41, number 2) of the *American Journal of Mathematics* contains the following papers: "Asymptotic satellites near the straight-line equilibrium points in the problem of three bodies," by DANIEL BUCHANAN; "Concerning the invariant theory of involutions of conics," by WAYNE SENSENIG; "Note on seminvariants of systems of partial differential equations," by A. L. NELSON; "On a method for determining the non-stationary state of heat in an ellipsoid," by BIBHUTIBHUSAN DATTA; "Nilpotent algebras generated by two units, i and j , such that i^2 is not an independent unit," by G. W. SMITH.

THE Circolo matematico di Palermo announces that it has resumed the publication of its *Rendiconti*. Contributors are invited to submit manuscripts to its committee of publication under the usual conditions.

BEGINNING with the volume for 1919, *Nyt Tidsskrift for Matematik* will be published by the Mathematical society of Copenhagen. The name of the periodical has been changed to *Matematisk Tidsskrift*.

AT the meeting of the London mathematical society held March 13, the following papers were read: By J. HAMMOND, "The solution of the quintic"; by L. J. MORDELL, "A simple algebraic summation of Gauss's sums"; by P. A. MACMAHON, "Divisors of numbers and their continuations in the theory of partitions"; by S. RAMANIJAN, "Convergence properties of partitions," and "Algebraic relations between certain infinite products."

AT the meeting of the Edinburgh mathematical society held March 14, the following papers were read: By C. G. KNOTT, "Systems of rays in quaternion symbolism"; by E. T. WHITTAKER, "Some disputed questions of probability."

THE British association for the advancement of science will resume the holding of summer meetings. In 1919 the meeting will be held at Bournemouth under the presidency of Sir CHARLES PARSONS.

THE Adams prize for 1918 has been awarded to Professor J. L. NICHOLSON, of King's College, London, for his researches in mathematical astronomy.

DR. G. WALLENBERG, of the Charlottenberg technical school, has been promoted to an honorary professorship of mathematics.

PROFESSOR O. HÖLDER has been chosen rector of the University of Leipzig.

ASSOCIATE professor W. MATTHIES, of the University of Basle, has been promoted to a full professorship of mathematical physics.

PROFESSOR G. ROST has been chosen rector of the University of Würzburg.

PROFESSOR J. K. LAMOND, of Pennsylvania College, Gettysburg, has been given two years leave of absence to engage in Red Cross work. He is now associate director of the department of military relief, in charge of home service, Pennsylvania-Delaware division.

DR. C. E. WILDER has been appointed instructor in mathematics at Clark University.

PROFESSOR W. H. GARNETT, of Wesleyan College, Winchester, Ky., is on leave of absence for the year.

PROFESSOR A. L. RHOTON, of the department of mathematics at Georgetown, Ky., College, has been appointed associate professor of education at Pennsylvania State College.

DR. G. W. MULLINS, of Barnard College, Columbia University, has been promoted to an assistant professorship of mathematics.

MR. W. W. ELLIOTT has been appointed instructor in mathematics at the University of Kentucky.

PROFESSOR H. G. ZEUTHEN, of the University of Copenhagen, died February 15, 1919, at the age of eighty years.

PROFESSOR F. E. MILLER, head of the department of mathematics at Otterbein College, died March 26, 1919. He had been a member of the American Mathematical Society since 1910.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BROUWER (L. E. J.). Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. 1ter Teil: Allgemeine Mengenlehre. Amsterdam, 1918. M. 2.00

BURNSIDE (W. S.) and PANTON (A. W.). The theory of equations. London, Longmans. Volume 1, 8th edition, 1918. 14 + 286 pp. Volume 2, 6th edition, 1916. 9 + 288 pp. Each, 10s. 6d.

FUETER (R.). Synthetische Zahlentheorie. Leipzig, Göschen, 1917. 8 + 271 pp.

- GOURSAT (E.). Cours d'analyse mathématique. Tome II: Théorie des fonctions analytiques. Equations différentielles. 3e édition. Paris, Gauthier-Villars, 1918. 8vo. 4 + 670 pp. Fr. 30.00
- KIEPERT (L.). Grundriss der Differential- und Integralrechnung. 2ter Teil: Integralrechnung. 11te vermehrte Auflage. Hanover, Helwing, 1918. M. 17.00
- . Tabelle der wichtigsten Formeln aus der Integralrechnung. Hanover, Helwing, 1918. M. 1.20
- KLEE (O.). Fermats letzter Satz. Berlin, Meyer und Müller, 1918.
- KOWALEWSKI (G.). Einführung in die Infinitesimalrechnung. 3te Auflage. Leipzig, Teubner, 1918. M. 1.20
- LIETZMANN (W.). Der pythagoreische Lehrsatz. Mit einem Ausblick auf das Fermatsche Problem. 2te Auflage. Leipzig, Teubner, 1918. 4 + 69 pp. M. 1.00
- LINDOW (M.). Differentialrechnung. (Aus Natur und Geisteswelt, Nr. 387.) 2te Auflage. Leipzig, Teubner, 1918. 83 pp. Geb. M. 1.50
- . Integralrechnung mit Aufgabensammlung. (Aus Natur und Geisteswelt.) 2te Auflage. Leipzig, Teubner, 1918. Geb. M. 1.50
- MEISSNER (O.). Wahrscheinlichkeitsrechnung. 2te Auflage. Leipzig, Teubner, 1918. M. 1.00
- MENDELSSOHN (W.). Einführung in die Mathematik. (Aus Natur und Geisteswelt, Nr. 503.) Leipzig, Teubner, 1918. 113 pp. Geb. M. 1.50
- PANTON (A. W.). See BURNSIDE (W. S.).
- PLANT (L. C.). See WEBBER (W. P.).
- PRÜSS (A.). Ueber vollständig zerfallende algebraische Raumkurven und Flächen. (Diss.) Kiel, 1918.
- ROSE (M.). Einleitung in die Funktionentheorie. Die komplexen Zahlen und ihre elementaren Funktionen. 2te verbesserte Auflage. Berlin, Göschen, 1918. M. 1.00
- SCHROECKH (C.). Die Höchstzahl der reellen Züge einer Raumkurve n ter Ordnung in m Dimensionen. (Diss.) Kiel, 1917.
- WEBBER (W. P.) and PLANT (L. C.). Introductory mathematical analysis. New York, Wiley, 1919. 13 + 304 pp. \$2.00
- WIELEITNER (H.). Der Begriff der Zahl in seiner logischen und historischen Entwicklung. (Mathematisch-physikalische Bibliothek.) 2te durchgesehene Auflage. Leipzig, Teubner, 1918. M. 1.00
- ZIEHEN (T.). Das Verhältnis der Logik zur Mengenlehre. Berlin, Reuther und Reichard, 1917.

II. ELEMENTARY MATHEMATICS.

- CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht. Leipzig, Teubner, 1918. 1ter Teil, 4 + 116 pp. 2ter Teil, 4 + 123 pp. M. 1.20 + 1.20
- FISCHER (P. B.). Rechenaufgaben aus der Kriegszeit. Leipzig, Teubner, 1917. 48 pp. M. 1.20
- HARTENSTEIN (F.). Fünfstellige logarithmische und trigonometrische Tafeln für den Schulgebrauch. 2te Auflage. Leipzig, Teubner, 1917.

- HEATH (R. S.). Solid geometry including mensuration of surfaces and solids. 4th edition. London, Rivingtons, 1918. 123 pp. 4s.
- JONES (F. T.). Problems and questions on algebra. Cleveland, University Supply and Book Company, 1919. 2 + 66 pp. \$0.40
- LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und Zahlenrechnen für Schule und Leben in neuer Anordnung. Leipzig, Teubner, 1918. 32 pp. M. 1.40
- RAUSCHELBACH (H.). Divisionstafel enthaltend drei- oder vierziffrige Quotienten aller ein- bis dreiziffrigen Dividenden und aller zweiziffrigen Divisoren. Göttingen, 1918. M. 7.60
- SCHMITZ (B.). Das neue Multiplikations-Verfahren. Leipzig, Adler, 1918. 13 pp. Geh. M. 2.00
- . "Triplex"; das neueste und vollkommenste Verfahren der amerikanischen Buchführung. Leipzig, Adler, 1918. 22 pp. + 4 plates. Geh. M. 3.00
- SCHUSTER (A.). Mathematische Unterrichtsbriefe. Leipzig, Verlag Naturwissenschaften, 1918. M. 6.45
- TIMERDING (H. E.). Der goldene Schnitt. (Mathematisch-physikalische Bibliothek, Band 32.) Leipzig, Teubner, 1918. M. 1.00

III. APPLIED MATHEMATICS.

- ANUARIO del observatorio de Madrid para 1919. Madrid, Imprenta de la Casa Editorial Bailly-Baillière, 1918. 741 pp.
- APPELL (P.) et DAUTHEVILLE (S.). Précis de mécanique rationnelle. Introduction à l'étude de la physique et de la mécanique appliquée. 2e édition. Paris, Gauthier-Villars, 1918. 8vo. 8 + 734 pp. Fr. 50.00
- AQUINO (R. DE). The "newest" navigation altitude and azimuth tables. 2d stereotyped edition enlarged and improved. London, Potter, 1918. 8vo. 52 + 176 + 40 pp. 12s.
- AUERBACH (F.). Die graphische Darstellung. 2te Auflage. Leipzig, Teubner, 1918. M. 1.20
- BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik für Metallindustrieschulen, vorzugsweise für Maschinenbauschulen (Werkmeisterschulen), die Unterstufe der höheren Maschinenbauschulen und verwandten technischen Lehranstalten. Bearbeitet von S. Jakobi und A. Schlie. 4te Auflage. Leipzig, Teubner, 1917. 6 + 223 pp. M. 2.80
- BLEICH (F.). Die Berechnung statisch unbestimmter Tragwerke nach der Methode des Viermomentensatzes. Berlin, 1918. M. 12.00
- BÖRNSTEIN (R.). Die Lehre von der Wärme. 2te Auflage. Leipzig, Teubner, 1918. M. 1.50
- BORTKIEWICZ (L. VON). Homogeneity and Stability in the Statistics. Upsala, Almquist und Wiksells, 1918.
- DAUTHEVILLE (S.). See APPELL (P.).
- DUNCAN (D. C.). See FERRY (E. S.).
- FERRY (E. S.). A handbook of physics measurements. In collaboration with O. W. Silvey, G. W. Sherman and D. C. Duncan. 2 volumes. New York, Wiley, 1918. 8vo. 9 + 251 + 10 + 233 pp. \$2.00 + \$2.00

- FÖPPL (A.). Vorlesungen über technische Mechanik. Band 3: Festigkeitslehre. 6te Auflage. Leipzig, 1918. M. 15.00
- GRAY (A.). A treatise on gyrostatics and rotational motion. Theory and applications. London, Macmillan, 1918. Super royal 8vo. 2£ 2s.
- HAUSSNER (R.). Darstellende Geometrie. 1ter Teil. 3te vermehrte und verbesserte Auflage. Berlin, 1918. M. 1.00
- HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. 1ter Teil. Berlin, 1918. M. 1.00
- HINRICKS (W.). Einführung in die geometrische Optik. Neudruck. Berlin, 1918. M. 1.00
- HUNTER (J. DE G.). The earth's axis and triangulation. (Survey of India, Professional paper No. 16.) Dehra Dun, Office of the Trigonometrical Survey, 1918. 8 + 219 pp. + 6 charts. 5s. 4d.
- JAKOBI (S.). See BARDEY (E.).
- LÖTZBEYER (P.). Darstellende Geometrie des Geländes. Dresden, Ehlermann, 1918.
- MANES (A.). Grundzüge des Versicherungswesens. 3te veränderte Auflage. Leipzig, 1918. M. 1.20
- MEISSNER (O.). Anwendungen der Wahrscheinlichkeitsrechnung. Leipzig, Teubner, 1918. M. 1.00
- MÜLLER (A.). Die Referenzflächen des Himmels und der Gestirne. Braunschweig, Vieweg, 1918. 162 pp. Geb. M. 6.60
- MÜLLER (R.). Leitfäden für die Vorlesung über darstellende Geometrie. 3te Auflage. Braunschweig, Vieweg, 1917. 179 pp. M. 7.00
- NORTON (A. P.). A star atlas and telescopic handbook (Epoch 1920) for students and amateurs. London and Edinburgh, Gall and Inglis, 1919. 25 pp. + 16 maps.
- POINCARÉ (H.). Die neue Mechanik. 3te Auflage. Leipzig, 1918. M. 1.00
- SCHLIE (A.). See BARDEY (E.).
- SCHUDEISKY (A.). Leitfäden für den neuzeitlichen Linearzeichenunterricht. Handbuch für den Lehrer. Leipzig, Teubner, 1916. 8 + 81 pp. M. 4.80
- SELME (L.). Principe de Carnot contre formule empirique de Clausius, essai sur le thermodynamique. Paris, Dunod et Pinat, 1917. 148 pp. Fr. 4.50
- SHERMAN (G. W.). See FERRY (E. S.).
- SILVEY (O. W.). See FERRY (E. S.).
- VATER (R.). Praktische Thermodynamik. (Aus Natur und Geisteswelt, Nr. 596.) Leipzig, Teubner, 1918. 96 pp. Geb. M. 1.50
- WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 2ter Band. Berlin, 1918. M. 12.00

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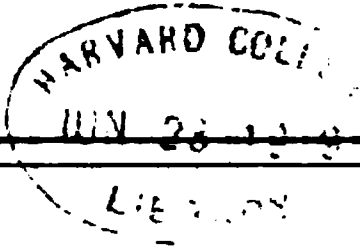
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The following dates have been fixed for the meetings of the Society :

Ann Arbor: September 2-4, 1919



BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

EDITED BY

F. N. COLE	VIRGIL SNYDER	J. W. YOUNG
ALEXANDER ZIWET	D. E. SMITH	
T. LEVI-CIVITA	R. C. ARCHIBALD	

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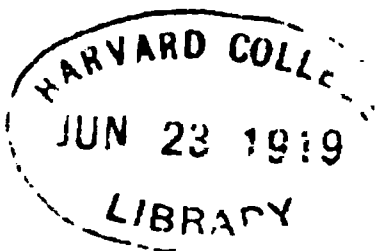
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THE MARCH MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE twelfth regular meeting of the American Mathematical Society at Chicago, being also the forty-third regular meeting of the Chicago Section, was held on Friday and Saturday, March 28 and 29, at the University of Chicago. The various sessions were attended by about forty persons, among whom were the following thirty-one members of the Society:

Dr. I. A. Barnett, Professor H. F. Blichfeldt, Professor G. A. Bliss, Dr. Henry Blumberg, Professor W. H. Bussey, Professor R. D. Carmichael, Professor E. W. Chittenden, Professor A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Professor L. W. Dowling, Mr. E. B. Escott, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Professor A. C. Lunn, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor H. L. Rietz, Professor W. H. Roever, Mr. J. B. Rosenbach, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. B. Van Vleck, Professor G. E. Wahlin, Professor E. J. Wilczynski, Professor J. W. A. Young, and Professor Alexander Ziwet.

Twenty-nine persons joined in a dinner at the Quadrangle Club on Friday evening. Following the dinner, a number of short informal speeches were made in response to calls from Professor Bliss who presided. Professor Slaught read interesting extracts from a letter which he had just received from Professor E. R. Hedrick, who is in France helping to organize educational work for members of the American Expeditionary Forces. Professor Ziwet of the University of Michigan extended a most cordial invitation to members of the Section to come to the summer session of the Society next September at Ann Arbor. Several members spoke against the too common practice among mathematicians of giving gratuitously information of a technical nature to persons untrained in mathematics; it was argued that the worth of mathematics would be more widely appreciated and its development correspondingly more rapid if mathematicians demanded adequate compensation for such services.

At a short business session on Saturday morning there was

a discussion of the desirability of holding the December meeting of the Section this year in St. Louis in conjunction with the meetings of the American association for the advancement of science. It was voted to leave the matter to the program committee for decision.

In reading his paper on "Differential corrections for anti-aircraft guns," Professor Bliss paid tribute to the efforts of mathematicians who have been engaged in war work of a mathematical nature, mentioning especially the effectiveness of the work headed by Major F. R. Moulton at Washington and by Major Oswald Veblen at the Aberdeen Proving Ground on ballistic problems.

Friday afternoon was devoted to a symposium on the geometry of numbers, relating largely to the work of Minkowski. Formal papers were presented as follows:

I. Professor H. F. BLICHFELDT: "Report on the theory of the geometry of numbers," giving the fundamental theorems with applications to homogeneous and non-homogeneous forms.

II. Professor L. E. DICKSON: "Applications of the geometry of numbers to algebraic numbers."

Synopses of these papers will appear in the July BULLETIN.

Professor G. A. Bliss, chairman of the Section, presided at the various sessions, being relieved by Professors Curtiss, Miller, and Carmichael. At the sessions on Friday and Saturday mornings the following papers were read:

(1) Professor E. J. WILCZYNSKI: "A new geometrical representation for a function of a complex variable."

(2) Dr. A. L. NELSON: "Plane nets conjugate to a given congruence of straight lines" (preliminary report).

(3) Professor A. B. COBLE: "On the ten nodes of a rational plane sextic and of the Cayley symmetroid."

(4) Dr. FLORENCE E. ALLEN: "On a class of sectrix curves."

(5) Professor W. H. BUSSEY: "A note on the problem of the eight queens."

(6) Professors E. W. CHITTENDEN and A. D. PITCHER: "On the theory of developments of an abstract class in relation to the calcul fonctionnel."

(7) Professor E. J. WILCZYNSKI: "The scientific work of Gabriel Marcus Green."

(8) Professor G. A. BLISS: "Differential corrections for anti-aircraft guns."

(9) Professor H. L. RIETZ: "Functional relations for which the coefficient of correlation is zero."

(10) Professor L. L. DINES: "Systems of linear inequalities."

(11) Professor E. J. MOULTON: "A note on a proof of the law of the mean."

(12) Dr. A. R. SCHWEITZER: "On pseudo-groups with an application to the algebra of logic."

(13) Professor G. A. MILLER: "Groups containing a relatively large number of operators of order two."

(14) Dr. HENRY BLUMBERG: "On a theorem of W. H. Young and G. C. Young."

(15) Professor E. B. VAN VLECK: "On Green's lemma."

In the absence of the authors, the papers of Dr. Nelson, Professor Coble, Dr. Allen and Professor Dines were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper, Professor Wilczynski interprets the complex variables z and w as points upon the same Neumann sphere, and joins by straight lines the points which correspond to each other by means of a functional relation $w = F(z)$. The resulting congruence has a large number of interesting properties. Its focal surfaces are real surfaces of positive curvature, and the foci on each line of the congruence divide harmonically its two points of intersection with the sphere. The congruence is a W -congruence, and all of the asymptotic lines of the focal surface belong to linear complexes. The Neuman sphere is the directrix quadric of the focal surface. The asymptotic lines of the focal surface correspond to the minimal lines of the z -plane and can be obtained with ease. The developables of the congruence, which are real, are also obtained and interesting relations are shown to exist between the directrices, axes, and rays of the two focal sheets, as well as between the directrix, axis, and ray curves.

2. Dr. Nelson has made use of Wilczynski's methods in the study of the projective properties of a congruence of straight lines as related to those of the plane nets which are conjugate to it, in the sense of Guichard. Following are some of the results.

(1) If the plane nets have straight lines for constant values of one parameter, one of the nappes of the focal surface of the congruence is developable, and conversely.

(2) If the first Laplace transforms of the plane nets have straight lines corresponding to the constant values of one parameter, the new focal nappe of the minus first derived congruence becomes developable, and conversely.

(3) If the plane nets are periodic under the Laplace transformation, of period three, the following properties hold for the congruence:

(a) The lines which join the first and second, and the minus first and minus second, Laplace transforms of the surface point of either of the focal nappes will intersect on the axis of that surface point.

(b) This point of intersection will be the harmonic conjugate of the surface point with respect to the foci of its axis. It will also be one of the foci of the joint axis which corresponds to the surface point.

(c) The joint axis curves and joint ray curves coincide and become parametric.

(d) The parametric tangents at a particular surface point of either focal nappe will be the double rays of the involution determined by the axis tangents and ray tangents at the same point.

3. The notion as developed by Professor Coble of the congruence of n points in a projective space under Cremona transformation leads to an infinite discontinuous group which can be exhibited as a group of linear transformations with integer coefficients. Thus the group has a modular theory and in an earlier paper the types of finite groups obtained by reduction modulo 2 were determined. It now appears that these modular groups have not merely an arithmetic but also a geometric existence when the n points are properly specialized. Such cases for example are the ten nodes of a rational plane sextic, and in space the ten nodes of a Cayley symmetroid. Some of the theorems obtained are as follows. A general rational plane sextic can be transformed into precisely $2^{13} \cdot 31 \cdot 51$ projectively distinct sextics and under such transformation these types are permuted by a group isomorphic with a theta modular group of genus 5. Again if there exists a quartic curve with a triple point at one node

and on the nine others (one condition on the sextic) then there exists a similar quartic with a triple point at *any* one of the nodes. Similar theorems apply to the symmetroid. In this case the genus of the group is 4. There is thus foreshadowed a uniform parametric representation of the ten nodes in terms of abelian modular functions of genus four.

4. A system of curves discussed in a paper presented to the Society by Dr. James H. Weaver in April, 1917, is shown by Miss Allen to be a special case of the Schoute sectrix curves and of the araneïd or "spider" curves. It is shown that the system for successive values of n may be generated by a birational transformation, consisting of an inversion followed by a perspectivity. The generation of the entire system of araneïds by alternate applications of these two operations is also proved.

5. The problem of the eight queens is the determination of the ways in which eight queens can be placed on a chess board—or more generally, in which n queens can be placed on a square board of n^2 cells—so that no queen can take any other queen. Professor Bussey's note shows that in the special case in which n is a prime number p , there are $p^2 - 3p$ solutions furnished by the $p^2 - 3p$ straight lines of slopes 2, 3, 4, $\dots$, $(p - 2)$ of the finite euclidean geometry of order p^2 . The solutions furnished by this method are not the only solutions, however.

6. In a paper, which has not been published, entitled "Classes which admit a development" (presented to the Society, March 21, 1913), Professors Chittenden and Pitcher discussed the theory of developments Δ of an abstract class in relation to the calcul fonctionnel of Fréchet. [A development Δ of a class P is a system $((P^{m_l}))$ of subclasses of P ($m = 1, 2, 3 \dots$; $l^m = l_1^m, l_2^m, \dots$). Cf. E. H. Moore, Introduction to a Form of General Analysis, Yale University Press, New Haven, 1910.] In the present paper the above mentioned theory is refounded and based upon a set of five properties of a development Δ which together imply that the class P is a compact metric space (cf. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914). The five properties are proved to be completely independent in the sense of E. H. Moore (loc. cit.).

The results of the general theory are applied to obtain the necessary and sufficient conditions that a topological space (cf. Hausdorff, loc. cit.) be equivalent to a compact metric space. A further application is made to spaces of non-metrical analysis situs. Every limited domain in a space δ satisfying the axiom system Σ_1 (Σ_2) of R. L. Moore (*Transactions of the American Mathematical Society*, volume 17, 1916) is a compact metric space. This result was obtained independently by R. L. Moore by another method.

The paper will appear in the *Transactions*.

7. Professor Wilczynski's paper, intended for publication in the BULLETIN, is devoted to a brief description of the mathematical work of Gabriel Marcus Green, whose premature death inflicts a heavy loss on American mathematics.

8. In the applications of ballistics it is customary to compute a first approximation to the path of a projectile on the assumption that there is no wind, and on the further assumptions that the density of the air and the weights of the projectile and powder charge are normal. These are ideal conditions rarely if ever realized in actual firing. The range table must therefore contain differential corrections which can be applied in the field, either mechanically or personally, to account for the actual conditions at the time when the firings are made. The so-called differential corrections appear as the solutions of a system of linear differential equations which are solved by mechanical quadratures. In the paper of Professor Bliss an auxiliary system of linear differential equations is used by means of which the number of solutions to be computed by mechanical quadratures is reduced to three. The remainder of the computation is of the simplest sort, requiring no especial skill, and relatively rapid.

9. In this paper, Professor Rietz treats some simple and striking cases in which the correlation coefficient of two variables x and y is zero although there is a simple functional relation between x and y . The result $r = 0$ may be accounted for by a consideration of the lines of regression. This brings out the importance of testing the curve of regression, and the danger of making applications of the correlation coefficient without knowledge of the underlying theory. In the cases

cited, for a single valued function $y = f(x)$, the correlation ratio η_y can be used appropriately instead of r to give a summary notion of correlation.

10. In this paper, Professor Dines constructs a theory relative to systems of linear inequalities, which to a considerable extent parallels the classic theory of systems of linear equations. The paper will appear in the *Annals of Mathematics*.

11. A proof of the law of the mean for derivatives has been based on the law of the mean for integrals (Goursat-Hedrick, *Mathematical Analysis*, volume I, page 155). Professor Moulton calls attention to an error in the proof, and shows how it can be corrected. The hypotheses of the theorem proved in this way are more restrictive than when proved in another common way. It is pointed out that if Denjoy's generalization of integration is used the derivative $f'(x)$ need not be continuous on the interval (a, b) , nor integrable in the sense of Riemann or Lebesgue. The restrictions are that $f'(x)$ must exist and be finite at every point of the interval.

12. A set E of elements forms a pseudo-group if E satisfies a set of properties $\Sigma[\rho_1, \rho_2, \dots]$ necessary for an abstract group (finite or infinite) and including closure with regard to the undefined relations $\rho_1, \rho_2, \dots$ generating Σ . Every group is a pseudo-group as are also the semi-groups of Dickson and Hilton. In a previous paper Dr. Schweitzer exhibited a pseudo-group $\Sigma(\theta)$ satisfied by $\theta(x, y) = x^{-1}y^{-1}$ of an abstract group. In the present paper Dr. Schweitzer constructs further pseudo-groups, in particular with reference to distributive properties. Among the latter class there exists a pseudo-group satisfied by $\zeta(x, y) = x^{-1}yx$ and $\tau(x, y) = xyx^{-1}$ of an abstract group and possessing duality with reference to the generating relations $\zeta(x, y)$ and $\tau(x, y)$. This pseudo-group $\Sigma[\zeta, \tau]$ has the property that it is satisfied by any set of elements obeying the "additive" and "multiplicative" laws of the algebra of logic. Incidentally, a new set of postulates for the latter discipline is obtained merely by assuming that for the pseudo-group $\Sigma[\zeta, \tau]$ the "principle of absorption" $\zeta[x, \tau(x, y)] = x$ is valid. Fundamental for the discussion is

the fact that the properties $\zeta(x, x) = x$, $\zeta[x, \tau(x, y)] = x$ or y , $\zeta[x, \tau(y, z)] = \tau[\zeta(x, y), \zeta(x, z)]$ and their duals are valid both for the abstract group $[\zeta(x, y) = x^{-1}yx, \tau(xy) = xyx^{-1}]$ and the algebra of logic $[\zeta(x, y) = x \cdot y, \tau(x, y) = x + y]$.

13. Professor Miller's paper appears in full in the present number of the BULLETIN.

14. A given planar set S is said to have a triangular void at a point P , if P is the vertex of a triangle containing no points of S in its interior. According to a recent result of W. H. and G. C. Young (*Proceedings of the London Mathematical Society*, 1918), a planar set having a triangular void at every one of its points is of measure zero. Dr. Blumberg gives a simple proof of this theorem, and a generalization based on the following definition: Let C_r be the circle of center P and radius r and $m_e(S, C_r)$ the exterior measure of the portion of S in C_r ; then we define the "upper exterior measure of S at P " as

$$\limsup_{r \rightarrow 0} \frac{m_e(S, C_r)}{\pi r^2}.$$

The generalization is as follows: A planar set having its upper exterior measure < 1 at every one of its points is of measure zero. This theorem is utilized to obtain certain general properties of functions of two variables. Extension to higher spaces is immediate.

15. In the proofs which have been given for Green's fundamental lemma

$$\int \int \frac{\partial P(x, y)}{\partial x} dx dy = \int_c P(x, y) dy$$

notable restrictions have been imposed upon the boundary. The usual condition is that the contour C shall be cut by a parallel to the axis of X in a finite number of points or consist of a finite number of "regular" pieces. The paper of Professor Van Vleck gave a proof for the general case in which C is merely conditioned to be a simple closed rectifiable curve. Over its closed interior $P(x, y)$ is supposed to be continuous, while the requirement for $\partial P / \partial x$ is that it shall be properly integrable.

E. J. MOULTON,

Acting Secretary of the Chicago Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-third regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 5, with the chairman, Professor Cajori, presiding. The total attendance was twenty, including the following thirteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor Thomas Buck, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Dr. Pauline Sperry, Dr. J. S. Taylor.

The next meeting of the Section will be held at the University of California, October 25, 1919.

The following papers were presented at the April meeting:

(1) Professor R. E. ALLARDICE: "Note on the sextic with eight cusps."

(2) Professor FLORIAN CAJORI: "Observations on the development of algebraic notations."

(3) Dr. J. S. TAYLOR: "A set of five postulates for Boolean algebras in terms of the operation 'exception.'"

(4) Dr. PAULINE SPERRY: "On the work of Gabriel Marcus Green in the field of projective differential geometry."

(5) Dr. Y. R. CHAO: "A note on 'continuous' mathematical induction."

(6) Professor L. M. HOSKINS: "Effect of the assumption of variable elasticity upon estimates of the rigidity of the earth."

(7) Professor W. A. MANNING: "A fundamental theorem for simply transitive primitive groups."

(8) Professor E. T. BELL: "On the number of representations of an integer as a sum of 3, 5, 7, 9, 11, or 13 squares."

(9) Professor E. T. BELL: "On the number of representations of $2n$ as a sum of $2r$ squares."

(10) Dr. L. L. SMAIL: "Summability of double series."

Dr. Chao was introduced by Professor Bernstein. The papers of Professor Bell and Dr. Smail were read by title. Abstracts of the papers follow.

1. In his note on the sextic with eight cusps, Professor Allardice finds the equation of this curve, and establishes the relations that connect the eight cuspidal points.

2. Professor Cajori observes that the adoption of algebraic notations is subject to two opposing forces: (1) The disinclination of mathematicians to memorize many symbols, (2) The convenience and power acquired from a well-chosen symbolism. In elementary algebra no large group of symbols invented by any one author has ever been bodily and permanently adopted; each symbol had its own struggle for existence, dependent not only upon merit but also upon particular configuration of circumstances. Bi-asymmetrical marks other than the numerals and letters of alphabets hardly ever survived. Notations have crossed national boundaries less easily than have principles and processes. There still exist side by side certain duplicate notations, rendering elementary algebra unnecessarily difficult. There is little precedent to indicate how much in the acquirement of desirable notations can be achieved by national and international organization.

3. Dr. Taylor presents a set of five comparatively simple postulates in terms of the operation "exception" with a proof of their consistency, independence, and sufficiency for the logic of classes. The complete existential theory of the five postulates is then developed, followed by a short discussion of the relative importance of the element 1 and negation (not- a). In order to show that each is as powerful as the other in one respect at least, a second set of postulates is presented in which, instead of postulating the element 1 and defining not- a , not- a is postulated and the element 1 defined, a thing which curiously enough has never been done heretofore.

4. In a brief résumé, Dr. Sperry recited some of Dr. Green's more important contributions to the theory of projective differential geometry, emphasizing especially his work on (a) the geometry of one-parameter families of space curves, and conjugate nets on a curved surface, and (b) the general theory of curved surfaces and rectilinear congruences, as best illustrating the uniqueness of his analytic methods and the keenness of his geometric intuitions.

5. Dr. Chao proves a theorem concerning a propositional function $\varphi(x)$ defined for a real interval and satisfying the following two conditions: (1) $\varphi(a)$ is true; (2) there exists a constant Δ in the interval such that $\varphi(x)$ implies $\varphi(x \pm \delta)$ for $0 < \delta \leq \Delta$.

6. Estimates of the rigidity of the earth are based upon the comparison of the actual yielding of the earth to tidal and centrifugal forces, as inferred from certain refined measurements, with the computed yielding of an elastic sphere having the dimensions and mass of the earth. The solution of this problem is known for a compressible sphere of uniform density, and for an incompressible sphere in which the density varies with distance from the center; the elastic moduli being in both solutions assumed to have uniform values throughout the body. Professor Hoskins has obtained the solution for a class of cases in which the elastic moduli as well as the density vary with distance from the center, and has obtained numerical results for a series of cases conforming closely with what is known regarding the actual properties of the earth. It is found that the known observational results are in close harmony with the computed strain of a sphere in which the elastic moduli are assumed to vary continuously from surface values based upon the known properties of rocks to values between six and seven times as great at the center. This would give a value of the modulus of rigidity at the center of about 1.70×10^{12} C. G. S. units—about double the value for steel.

7. The theorem stated by Professor Manning is to this effect: If the subgroup that leaves one letter of a simply transitive primitive group fixed has a multiply transitive constituent, of degree m say, then one of two things must occur; either all the transitive constituents of the subgroup that fixes one letter are multiply transitive groups of the same degree m in simple isomorphism, or at least one transitive constituent of the maximal subgroup in question is of a degree greater than m and a divisor of $m(m - 1)$.

8. Defining as a primitive function of a positive integer n one whose values may be calculated from the real divisors of n alone, and as a simple function any sum of a finite number of primitive functions whose arguments form a recurring

series of the second order, Professor Bell determines all cases in which the number of representations of an integer as a sum of 3, 5, 7, 9, 11, or 13 squares is simple. The integers to be represented are classified first according to their linear forms modulo 8. For 3, 5, 7, 9 squares all possible cases are simple; for 11 or 13 squares only some are simple, and for 15, 17, 19, 21, 23, 25 squares none are simple. The criteria deciding this seem to indicate non-simplicity for all odd numbers of squares > 25 ; hence a special interest attaches to the cases treated in the paper. The resulting formulas are well adapted to numerical computation, and a comparison with the existing formulas for 5 or 7 squares (Eisenstein, H. J. S. Smith) shows a marked saving of labor in actual use. For 3, 5 and 7 squares, and for some cases of 9, 11 or 13, the number of representations may be calculated by several rapid recurrences. An interesting by-product of the determination for 3 squares is a new derivation of the class number for a negative determinant by finite processes only. This leads to results, in finite form, which also easily yield numerical values, and which should be of use in the construction of tables beyond those existing, should the necessity for such arise.

9. In two papers Liouville (*Journal de Mathématiques*, series 2, volume 6, 1861, pages 233,369) gave four general formulas relating to the number of representations of $2n$ as a sum of $2r$ squares. These contain as very special cases all the known instances in which such representations are either primitive or simple as defined in the preceding paper. Liouville suppressed both the proofs and the essential detail of showing how the undetermined coefficients, without knowledge of which his formulas cannot be used, may be found; these are supplied in Professor Bell's second paper. In proving the formulas it appears that similar general theorems exist for $2n + 1$ as a sum of $2r$ squares, and that corresponding but less elegant results may be stated for both $2n$ and $2n + 1$ as a sum of $2r + 1$ squares. Total and proper (as defined by Eisenstein) representations are treated by one analysis.

10. In a recent paper in the *Annals of Mathematics* (December, 1918), Dr. Smail has given a general method of summation for divergent series, based on a summation formula involving a function undefined except for certain restrictive

conditions. In the present paper he extends this method to double series, and discusses the application of the method to convergent double series. It is found that the summation functions of the familiar methods of Cesàro, Hölder, Borel, LeRoy, Riesz, Vallée-Poussin, etc., can be used in building up summation formulas for double series.

B. A. BERNSTEIN,
Secretary of the Section.

ON A CERTAIN GENERATION OF RATIONAL CIRCULAR AND ISOTROPIC CURVES.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society December 28, 1917.)

1. *Introduction.*

A CIRCULAR curve contains the circular points at infinity, or the isotropic points of the plane, as single, or as singular points. A plane isotropic curve is defined as a curve, all of whose infinite points are absorbed by the isotropic points. The equation of such a curve, which is necessarily of even order, in cartesian coordinates may be written in the form

$$(1) \quad (x^2 + y^2)^k + \varphi(x, y) = 0,$$

in which $\varphi(x, y)$ is a polynomial of degree $2k - 1$ at most.

If $P(\xi, \eta)$ is a fixed point and $A(x, y)$ any other point so that $PA = \rho$, and θ the angle between PA and the positive direction of the x -axis, then the coordinates of A are $x = \xi + \rho \cos \theta$, $y = \eta + \rho \sin \theta$, and satisfy equation (1) when A is on the curve. The condition for this is an equation of the form

$$(2) \quad \rho^{2k} + \alpha_1 \rho^{2k-1} + \alpha_2 \rho^{2k-2} + \cdots + \alpha_{2k-1} \rho + \alpha_{2k} = 0,$$

in which $\alpha_1, \alpha_2, \cdots, \alpha_{2k-1}$ are coefficients, which, in general, depend on ξ, η, θ and the coefficients of (1); while α_{2k} is independent of θ . The roots $\rho_1, \rho_2, \cdots, \rho_{2k}$ of (2) are the distances PA_i ($i = 1, 2, 3, \cdots, 2k$) of the points of inter-

section $A_1, A_2, \dots, A_{2k}$ of the transversal through P , including an angle θ with the positive part of the x -axis. Evidently the product

$$(3) \quad \rho_1 \rho_2 \cdots \rho_{2k} = \alpha_{2k}$$

is constant for all transversals through P .

This constant product is called the power of the point P with respect to the isotropic curve. This is one of the principal properties* of an isotropic curve. In fact an isotropic curve may also be defined by this property. In this paper I shall establish the necessary and sufficient conditions for the form of parametric representation of rational circular, in particular of rational isotropic curves, and their generation by rational transformations in a complex plane. As an application the complete representation of all rational circular cubics, and all rational isotropic quartics will be given.

2. Parametric Representation of Rational Circular and Isotropic Curves.

Let

$$(4) \quad x = \frac{F(t)}{H(t)}, \quad y = \frac{G(t)}{H(t)},$$

in which F, G, H are polynomials in t , with real coefficients, without a common factor, represent a rational curve. To real values of t correspond real values of x and y , or real points on the curve. The curve passes once through each of the isotropic points, when for a complex value t_i of t

$$(5) \quad \lim_{t \rightarrow t_i} (x) = \infty, \quad \lim_{t \rightarrow t_i} (y) = \infty, \quad \lim_{t \rightarrow t_i} \left(\frac{y}{x} \right) = +i,$$

and, as a consequence, also for the conjugate value $\bar{t}_i$

$$(6) \quad \lim_{t \rightarrow \bar{t}_i} (x) = \infty, \quad \lim_{t \rightarrow \bar{t}_i} (y) = \infty, \quad \lim_{t \rightarrow \bar{t}_i} \left(\frac{y}{x} \right) = -i.$$

The algebraic sign of i might, of course, be reversed. From this it follows that a necessary condition that the curve (4) pass through the isotropic points is that $H(t) = 0$ has complex roots which are not common to $F(t) = 0$ and $G(t) = 0$. But

* E. Pascal, Repertorium der höhern Mathematik, vol. 2 (first half), pp. 436-438 (1910).

complex roots of $H(t) = 0$ occur in conjugate pairs. Let $t_1, \bar{t}_1; t_2, \bar{t}_2; \dots; t_k, \bar{t}_k$ be complex roots of this equation, so that $H(t)$ may be written in the form

$$(7) \quad H(t) = [(t - t_1)(t - t_2) \cdots (t - t_k)][(t - \bar{t}_1)(t - \bar{t}_2) \cdots (t - \bar{t}_k)]\Phi(t),$$

where $\Phi(t)$ is a polynomial in t , which contains none of the other factors of $H(t)$. The two brackets in (7) may be written in the form

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k,$$

and

$$t^k + \bar{\alpha}_1 t^{k-1} + \bar{\alpha}_2 t^{k-2} + \cdots + \bar{\alpha}_k,$$

and are clearly conjugate expressions, so that the first may be written in the form $r - is$, the second in the form $r + is$, in which r and s are in general polynomials in t of degree k . Hence, (7) has the form

$$(8) \quad H(t) = (r^2 + s^2)\Phi(t).$$

Incidentally we have proved

THEOREM 1: *If a polynomial equation with one unknown, and real coefficients, has only imaginary roots, then the polynomial may be represented as the sum of the squares of two other polynomials.*

Suppose now that t_i satisfies the equation $r - is = 0$, and that for this value of t not both r and s vanish simultaneously. Then t_i , according to (5), defines an isotropic point of the curve, when t_i is a root of the equation

$$(9) \quad G(t) - iF(t) = 0,$$

or

$$F(t) + iG(t) = 0.$$

Consequently, when the roots $t_1, t_2, \dots, t_k$ of $r - is = 0$ all define isotropic points, (9) may be written in the form

$$(10) \quad F(t) + iG(t) = (r - is)\Psi(t),$$

where $\Psi(t)$ is a polynomial in t , which in general, has complex coefficients, and which, when u and v denote polynomials in t with real coefficients, may be written in the form

$$\Psi(t) = u + iv.$$

u and v must not vanish identically simultaneously, and may reduce to constants. Now

$$(r - is)(u + iv) = ru + sv + i(rv - su),$$

so that from (10)

$$(11) \quad \begin{aligned} F(t) &= ru + sv, \\ G(t) &= rv - su. \end{aligned}$$

Hence

THEOREM 2: *Every rational circular curve may be parametrically represented in the form*

$$(12) \quad \begin{aligned} x &= \frac{ru + sv}{(r^2 + s^2)\Phi}, \\ y &= \frac{rv - su}{(r^2 + s^2)\Phi}, \end{aligned}$$

where all letters on the right hand side stand for polynomials in t , such that r and s have no common roots. The polynomials $ru + sv$, $rv - su$, $(r^2 + s^2)\Phi$, likewise have no common roots. When the degree of $r^2 + s^2$ is $2k$, then the curve has each of the circular points as a k -fold point. Conversely, from (12) follows readily that every parametric representation as defined by (12) represents a circular curve with the circular points as k -fold points.

When Φ reduces to a constant, which we may place equal to 1, and the degrees of $ru + sv$ and $rv - su$ are equal to or less than that of $r^2 + s^2$, all infinite points of the curve are at the circular points, so that, in this case, the curve is isotropic.

3. Isotropic Curves in a Complex Plane.

From the parametric representation of an isotropic curve

$$(13) \quad x' = \frac{ru + sv}{r^2 + s^2}, \quad y' = \frac{rv - su}{r^2 + s^2},$$

we find

$$(14) \quad x' + iy' = \frac{u + iv}{r + is}.$$

If we replace t in u , v , r , s by the complex variable

$$t = z = x + iy,$$

and put $x' + iy' = z'$, (14) assumes the form

$$(15) \quad z' = f(z)/g(z),$$

in which $f(z) = u + iv$, $g(z) = r + is$, and $f(z)$ is a polynomial whose degree is at most equal to that of $g(z)$, and has no factor in common with $g(z)$. Moreover $g(z)$ has no real roots. Hence

THEOREM 3: *By the rational transformation*

$$z' = f(z)/g(z),$$

in which $f(z)$ and $g(z)$ are polynomials as defined above, the real axis of the z -plane is transformed into a rational isotropic curve. On the other hand, when $f(z)$ and $g(z)$ are arbitrarily given in advance, subject to the conditions that the degree of $f(z)$ is at most equal to that of $g(z)$, that $g(z) = 0$ has no real roots, and $f(z) = 0$ and $g(z) = 0$ have no common roots, then (15) always defines an isotropic curve.

The proof of the second part of the theorem follows easily by deriving equations (14) and (13) from (15) in the reversed order.

Instead of separating in $f(z)$ and $g(z)$ the terms with real coefficients from those with complex coefficients, as understood in (14), we can also separate all real from all imaginary terms, when considering $f(z)$ and $g(z)$ as polynomials in x and y . Thus, from (15) we get an expression for z' of the same form as (14), but in which u, v, r, s are now real polynomials in x and y , which satisfy the Riemann-Cauchy differential equations. Formulas (13), with the new meaning of u, v, r, s , represent the same isotropic curve as before, if we let $y = 0$, and if we let $x = t$ assume all values of the real axis.

When in (15) z describes the real axis, the cartesian equation of the corresponding isotropic curve described by z' is obtained in the following manner: For the real axis $z = \bar{z}$; consequently, when

$$f(z) = z^m + a_1 z^{m-1} + a_2 z^{m-2} + \dots + a_m,$$

$$g(z) = b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n,$$

with $m \leq n$, we have the conditional equations,

$$(16) \quad \begin{aligned} & z'(b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n) \\ & \quad - (z^m + a_1 z^{m-1} + a_2 z^{m-2} + \dots + a_m) = 0 \\ & z'(\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \bar{b}_2 z^{n-2} + \dots + \bar{b}_n) \\ & \quad - (z^m + \bar{a}_1 z^{m-1} + \bar{a}_2 z^{m-2} + \dots + \bar{a}_m) = 0. \end{aligned}$$

The resultant of these two equations in z is a polynomial in z' and $\bar{z}'$, which, when set equal to zero, and on replacing z' by $x' + iy'$, $\bar{z}'$ by $x' - iy'$, reduces to the required equation of the isotropic curve. As both equations in (16) are of degree n , the degree of this curve will, in general, be $2n$, which is also apparent from (13). This method of finding the curve described by z' , when z describes the real axis, is valid for any rational transformation between z and z' , and includes the generation of all rational circular curves.

4. Rational Circular Cubics.

As an example for this method, the cartesian equation of all rational circular cubics, with the origin as a singular point, will be derived from the corresponding rational transformation of the superposed complex plane

$$(17) \quad z' = \frac{z^2 + az + b}{(\alpha + i\beta)z + (\gamma + i\delta)},$$

where $a, b, \alpha, \beta, \gamma, \delta$ are real, and $\alpha\delta - \beta\gamma \neq 0$. Equations (16) have now the form

$$(18) \quad z^2 + [a - (\alpha + i\beta)z']z + b - (\gamma + i\delta)z' = 0,$$

$$(19) \quad z^2 + [a - (\alpha - i\beta)\bar{z}']z + b - (\gamma - i\delta)\bar{z}' = 0.$$

Eliminating z between (18) and (19), the resulting equation in the cartesian plane reduces to

$$(20) \quad \begin{aligned} &(\alpha\delta - \beta\gamma)(x'^2 + y'^2)(\beta x' + \alpha y') + (\delta^2 - a\beta\delta + b\beta^2)x'^2 \\ &+ (2\gamma\delta - a\alpha\delta - a\beta\gamma + 2b\alpha\beta)x'y' \\ &+ (\delta^2 - a\alpha\gamma + b\alpha^2)y'^2 = 0. \end{aligned}$$

This may represent any rational circular cubic, with the singular point at the origin, by choosing $a, b, \alpha, \beta, \gamma, \delta$ properly.

5. Rational Bicircular Quartics.

Instead of treating quartics in the same manner as circular cubics, the reversed order will here be followed. The problem now is, to find the rational transformation in the complex plane, when the equation of the curve is given. The equation of any bicircular rational quartic, with the origin as a singular

point, may be written in the form

$$(21) \quad (x'^2 + y'^2)^2 + (\alpha x' + \beta y')(x'^2 + y'^2) \\ - (p^2 + \alpha p)x'^2 + bx'y' + cy'^2 = 0,$$

in which p is the distance from the origin of one of the four points of intersection of the quartic with the x' -axis.

The circle

$$(22) \quad x'^2 + y'^2 - px' - 2ty' = 0,$$

through this point and the origin, cuts the quartic in only one variable point, corresponding to the parameter t . The coordinates of (21) can therefore be expressed rationally in terms of t , by solving (21) and (22) simultaneously. The result is

$$(23) \quad x' = \frac{p(4t^2 + 2\beta t + c)^2 - 2t(4pt + 2\alpha t + \beta p + b)(4t^2 + 2\beta t + c)}{(4t^2 + 2\beta t + c)^2 + (4pt + 2\alpha t + \beta p + b)^2},$$

$$(24) \quad y' = \frac{2t(4pt + 2\alpha t + \beta p + b)^2 - p(4t^2 + 2\beta t + c)(4pt + 2\alpha t + \beta p + b)}{(4t^2 + 2\beta t + c)^2 + (4pt + 2\alpha t + \beta p + b)^2}.$$

If we put $4t^2 + 2\beta t + c = r$, $4pt + 2\alpha t + \beta p + b = s$, (23) and (24) may be written in the form

$$(25) \quad x' = \frac{ru}{r^2 + s^2},$$

$$(26) \quad y' = \frac{-su}{r^2 + s^2},$$

which is in agreement with the general result contained in (13), in which $v = 0$. Hence

$$z' = \frac{u}{r + is},$$

or, putting $t = z$,

$$(27) \quad z' = \frac{-4(\alpha + p)z^2 - 2bz + cp}{4z^2 + [2\beta + i(4p + 2\alpha)]z + i(\beta p + b)}$$

is the required transformation, which transforms the real axis into the given rational bicircular quartic.

For example, Bernoulli's lemniscate

$$(28) \quad (x'^2 + y'^2)^2 - 2a^2(x'^2 - y'^2) = 0$$

is obtained by the transformation

$$(29) \quad z' = \frac{a\sqrt{2}(z^2 - 1)}{-z^2 - 2iz - 1}.$$

6. Transformation of a General Algebraic Curve in the z -Plane.

Instead of restricting ourselves to the rational transformation of the real axis in the z -plane, we may ask the question, what effect the transformation (13) has upon an algebraic curve C of the order μ and of deficiency p , when u, v, r, s are obtained as polynomials of x and y from the rational transformation (15), in which the degree of $g(z)$ is n , that of $f(z)$ m , with $m \leq n$. When C does not pass through the common intersections of the curves $r = 0$ and $s = 0$, C will intersect the curve $r^2 + s^2 = 0$ in $2\mu n$ imaginary points which by (13) are transformed into the circular points at infinity. Each circular point is therefore a μn -fold point of the transformed curve C' . Now to a straight line l' in the z' -plane corresponds in the z -plane a curve l of order $2n$. This curve l cuts C in $2\mu n$ points, which, conversely, are transformed into the $2\mu n$ intersections of l' with C' . The order of C' is therefore $2\mu n$, and as the circular points absorb all $2\mu n$ infinite points of C' , this curve will be an *isotropic curve*. According to Clebsch,* the deficiency of C' is the same as that of C , so that a rational curve C is transformed into a rational isotropic curve C' . Evidently nothing in the generality of representation of rational circular and isotropic curves is lost by taking in place of a general curve C , the real axis. It is therefore not necessary, for our purpose, and in this place, to give further details of the relation between C and C' .

* Vorlesungen über Geometrie, vol. 1, pp. 661-674 (1876).

THE SELF-DUAL PLANE RATIONAL QUINTIC.

BY PROFESSOR L. E. WEAR.

A SELF-DUAL curve is defined to be a curve which has the same number of cusps and double points as it has inflexional tangents and double tangents respectively; and furthermore there are correlations—including polarities—which send the curve into itself.

Haskell, in this BULLETIN, January, 1917, found the maximum number of cusps of an algebraic plane curve, and enumerated the self-dual curves. The well known binomial curves

$$x_1^n = x_0^{n-r} x_2^r$$

have been extensively studied and shown to be self-dual.* The case of the rational plane quartic has been considered in my dissertation at the Johns Hopkins University.†

We here consider briefly the quintic. Since the class of the curve is to equal the order, we have as the fundamental equation,

$$n = n(n-1) - 2d - 3c,$$

where d is the number of double points and c the number of cusps. Hence we have for the quintic,

$$2d + 3c = 15,$$

an equation which has three solutions, as follows:

- (1) $d = 0, \quad c = 5,$
- (2) $d = 3, \quad c = 3,$
- (3) $d = 6, \quad c = 1.$

Case (3) may arise from the degenerate quintic composed of a conic and a cuspidal cubic.

The second case, that of the rational quintic, is the one to be considered here. Furthermore, we consider the curve which

* Loria, *Spezielle Ebene Kurven*, p. 308; Wieleitner, *Algebraische Kurven*, p. 136; Snyder, *American Journal*, vol. 30; Winger, *American Journal*, vol. 36.

† "The self-dual plane rational quartic," Dissertation, Johns Hopkins University, May, 1913.

is self-dual in all possible ways, which will be invariant under the largest possible group of transformations. The cusps then will be distinct. By taking the products of the correlations two at a time we obtain the collineations of a collineation group under which the curve is self-projective. These must interchange cusps, say, in all possible ways, and hence the curve must be invariant under a G_6 composed of a cyclic g_3 , the elements of which interchange the cusps cyclically, and three elements, obtained by adding to the g_3 an element of period two, and which leave one cusp fixed while interchanging the other two.

Now the equations of the rational quintic invariant under the dihedral G_6 are*

$$(1) \quad x_0 = t^5 + 5t^2, \quad x_1 = 5t^3 + 1, \quad x_2 = t^4 + t.$$

The flexes are $t^3 + 1$, and the cusps $t^3 - 1$. The G_6 is generated by the elements

$$(2) \quad t' = \omega t, \quad t' = 1/t, \quad (\omega^3 = 1),$$

with the appropriate ternary transformations

$$(3) \quad x_0' = x_0, \quad x_1' = \omega x_1, \quad x_2' = \omega^2 x_2.$$

$$(4) \quad x_0' = x_1, \quad x_1' = x_0, \quad x_2' = x_2.$$

Let us now add a correlation which will send any point of the curve into a line of the curve, and vice versa. In particular we desire a correlation that will interchange cusps and flexes, and likewise double points and double lines.

In order to obtain the correlation we need the line equations of the curve, which are obtained by taking the Jacobians of (1) two at a time, and are

$$(5) \quad \xi_0 = 5\tau^3 - 1, \quad \xi_1 = \tau^2(\tau^3 - 5), \quad \xi_2 = -10\tau(\tau^3 - 1).$$

The binary transformation $t\tau = -1$ will send the cusp $t = 1$ into the flex line $\tau = -1$, and conversely. Let us ask that this send *any point* of the curve into a *line* of the curve. Now any point of the curve is

$$(6) \quad \xi_0(t^5 + 5t^2) + \xi_1(5t^3 + 1) + \xi_2(t^4 + t) = 0$$

and any line is,

$$(7) \quad x_0(5\tau^3 - 1) + x_1(\tau^3 - 5) - x_2 10\tau(\tau^3 - 1) = 0.$$

* See Winger, l. c., p. 73.

Making the substitution $t = -1/\tau$ in (6), we have, after simplifying,

$$(8) \quad \xi_0(5\tau^3 - 1) + \xi_1(\tau^5 - 5\tau^2) + \xi_2(\tau - \tau^4) = 0.$$

By identifying (7) with (8) there results the polarity

$$(9) \quad \xi_0 = x_0, \quad \xi_1 = x_1, \quad \xi_2 = 10x_2.$$

Combining (9) with the elements of the collineation group G_6 , we obtain altogether six correlations which leave the curve unaltered. The latter statement is true since elements of the G_6 send any point of the curve into a second point, and this transformation followed by (9) must send the original point into a line of the curve. The correlations, with the elements of the G_6 , make up a G_{12} of collineations and correlations under which the curve is invariant. The following table gives the elements of the group, binary and ternary:

Collineations.

	x_0'	x_1'	x_2'	t'
1:	$= x_0,$	$= x_1,$	$= x_2,$	$= t.$
S :	$x_0,$	$\omega x_1,$	$\omega^2 x_2,$	$\omega t.$
S^2 :	$x_0,$	$\omega^2 x_1,$	$\omega x_2,$	$\omega^2 t.$
T :	$x_1,$	$x_0,$	$x_2,$	$1/t.$
ST :	$x_1,$	$\omega x_0,$	$\omega^2 x_2,$	$\omega/t.$
S^2T :	$x_1,$	$\omega^2 x_0,$	$\omega x_2,$	$\omega^2/t.$

Correlations.

	ξ_0	ξ_1	ξ_2	τ
Π_0 :	$= x_0,$	$= x_1,$	$= 10x_2,$	$= -1/t.$
$\Pi_0 S$:	$x_0,$	$\omega x_1,$	$10\omega^2 x_2,$	$-\omega/t.$
$\Pi_0 S^2$:	$x_0,$	$\omega^2 x_1,$	$10\omega x_2,$	$-\omega^2/t.$
$\Pi_0 T$:	$x_1,$	$x_0,$	$10x_2,$	$-t.$
$\Pi_0 \cdot (ST)$:	$x_1,$	$\omega x_0,$	$10\omega^2 x_2,$	$-\omega^2 t.$
$\Pi_0 \cdot (S^2T)$:	$x_1,$	$\omega^2 x_0,$	$10\omega x_2,$	$-\omega t.$

It is easily verifiable that these elements have the group properties. Only the first four of the correlations are polarities, and of these $\Pi_0 T$ alone refers to a real conic, the equation of which is

$$x_0 x_1 + 5x_2^2 = 0.$$

This conic is tangent to the curve at $t = 0$, $t = \infty$, and intersects the curve at six other points. At one of the latter points a tangent to the conic is tangent to the curve at some other point. We may summarize with this theorem: *The self-dual plane rational quintic admitting of the greatest possible number of correlations is invariant under a G_{12} consisting of collineations and correlations.*

THROOP COLLEGE,
February, 1919.

GROUPS CONTAINING A RELATIVELY LARGE NUMBER OF OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society March 29, 1919.)

§ 1. *Introduction.*

It is well known that every group which contains at least one operator of order 2 must contain an odd number of such operators and that there is an infinite number of groups such that each of them contains exactly $2m + 1$ operators of order 2, where m is an arbitrary positive integer or 0. It is also known that if exactly one half of the operators of a group are of order 2 then the order of this group must be of the form $2(2m + 1)$ and it must be the dihedral or the generalized dihedral group of this order. Moreover, it has been proved that a group G of order

$$g = 2^\alpha(2m + 1)$$

cannot contain more than $2^\alpha m + 2^\alpha - 1$ operators of order 2, α being an arbitrary positive integer, and whenever G contains this number of operators of order 2 it is either the abelian group of order 2^α and of type $(1, 1, 1, \dots)$ or it is the direct product of the abelian group of order $2^{\alpha-1}$ and of type $(1, 1, 1, \dots)$ and the dihedral or the generalized dihedral group of order $2(2m + 1)$.*

* G. A. Miller, this BULLETIN, vol. 13 (1907), p. 235.

From these theorems it results directly that when the order of a group is given but no other restrictions are imposed on the group, it is always possible to find an integer which represents the upper limit for the number of operators of order 2 contained in a group of this order, and also to state how many groups of this order contain this number of operators of order 2. In particular, there is no group of order 1,000 which contains more than

$$2^3 \cdot 62 + 2^3 - 1 = 503$$

operators of order 2, and there are exactly three groups of order 1,000 which contain separately 503 operators of this order since there are three abelian groups of order p^3 , p being a prime number.

When the number of operators of order 2 contained in G exceeds $g/2$, this excess cannot be an even number, for if it were an even number $2m$ it would result that the order of G would have to be twice an odd number. In fact, if we let $2k$ represent the number of the operators of order greater than 2 in such a group, it would follow that

$$2k + 2m + 1 = g/2.$$

Since a group whose order is twice an odd number contains a subgroup of half its order composed of its operators of odd order, it results that $m = 0$ whenever $2m$ represents the number of the operators of order 2 in excess of half the order of the group. That is, *whenever more than half of the operators of a group are of order 2 this excess is an odd number.* This elementary theorem will be generalized in the following section.

Let $g/2 - k$, k being a positive integer, represent the number of the operators of order 2 contained in G . When k is even, g is of the form $2(2m + 1)$ and hence G contains a subgroup of order $2m + 1$. If t represents any operator of order 2 contained in G , the product of t and an operator in this subgroup of order $2m + 1$ cannot be of order 2 unless t transforms this operator into its inverse. As all of the operators of a group must correspond to their inverses whenever more than three-fourths of them correspond to their inverses in an automorphism of the group, it results that when k is even $2m + 1 \leq 4k$. In particular, *there is only a finite number of groups which satisfy the condition that the number of their operators of order 2 is equal to half the order of the group minus a given even number.*

While the number of the operators of order 2 contained in a group cannot be equal to one half of the order of the group plus a positive even number, it can be equal to one-half the order of the group minus an arbitrary positive number. In fact, there is a cyclic group such that the number of its operators of order 2 is $g/2 - k$, k being an arbitrary positive integer. The order of this cyclic group is clearly $2(k + 1)$. When $k=2$ or 4 there is no other group satisfying the given condition, but when $k = 6$ there is also a non-cyclic group of order 18 which involves exactly $9 - 6 = 3$ operators of order 2, as can easily be verified.

§ 2. *Groups in Which More Than One-Half of the Operators are of Order Two.*

Whenever more than one half of the operators of G are of order 2, this excess must be an odd number, as was noted above. We shall now prove that this odd number is always of the form $2^n - 1$. When G is abelian and of type $(1, 1, 1, \dots)$ it is evident that this condition is satisfied. In all other cases G contains a non-invariant operator s_1 of order 2. Let H_1 represent the subgroup of G composed of all the operators of G which are commutative with s_1 and let $G - H_1$ represent the totality of the operators of G which are not contained in H_1 . Since each of the operators of $G - H_1$ is non-commutative with s_1 it results that at least one half of these operators have orders which exceed 2, and hence more than one half of the operators of H_1 are of order 2.

When H_1 is abelian it must be of order 2^n and of type $(1, 1, 1, \dots)$. If it is non-abelian, it contains a non-invariant operator s_2 of order 2, and we let H_2 represent the subgroup composed of all the operators of H_1 which are commutative with s_2 . The totality of operators $H_1 - H_2$ will again contain at least as many operators whose orders exceed 2 as the number of its operators of order 2, and the central of H_2 must exceed that of H_1 , which, in turn, exceeds that of G . By continuing this process we must arrive at an abelian group H_m composed of all the operators of H_{m-1} which are commutative with one of its non-invariant operators s_m of order 2. The subgroup H_m has an order of the form 2^n and is of type $(1, 1, 1, \dots)$.

Since H_m is a subgroup of G it is well known that all the operators of G may be uniquely represented as follows:

$$H_m + H_mt_2 + H_mt_3 + \dots + H_mt_r.$$

As H_m contains each of the operators $s_1, s_2, \dots, s_m$, it is evident that at least half of the operators in each of these co-sets have orders which exceed 2. On the other hand, there is at most one of the co-sets $H_mt_2, H_mt_3, \dots, H_mt_\gamma$ in which the number of operators whose orders exceeds 2 is larger than the number of its operators of order 2. To prove this fact it is only necessary to observe that the number of operators of order 2 in Ht_α , $2 \leq \alpha \leq \gamma$, is equal to the order of the subgroup of H_m composed of its operators which are commutative with t_α . If this order were less than 2^{n-1} for two values of α the number of operators of order 2 contained in G would be less than $g/2$, 2^n being the order of H_m .

If the number of operators of order 2 in each of the co-sets $H_mt_2, H_mt_3, \dots, H_mt_\gamma$ is equal to 2^{n-1} then 2^{n-1} represents also the excess over $g/2$ of the number of the operators of order 2 contained in G . If one of these co-sets Ht_α contains more operators whose orders exceed 2 than operators of order 2 this excess is equal to the number of operators of order 2 which are both contained in H_m and commutative with t_α . Hence it has been proved that *whenever the number of the operators of order 2 contained in a group exceeds one half of the order of the group this excess must be of the form $2^a - 1$.*

From the theorem which has just been proved it is easy to find the form of all the possible ratios between g and the number of operators in G whose orders exceed 2 whenever g is of the form 2^k . In fact, this number is evidently $2^{k-1} - 2^a$ and hence this ratio is always of the form

$$\frac{2^k}{2^{k-1} - 2^a} = \frac{2^\beta}{2^{\beta-1} - 1}.$$

Moreover, there is an infinite system of such groups for every positive integral value of $\beta > 1$.* It may also be noted that whenever one of the co-sets Ht_α , $2 \leq \alpha \leq \gamma$, involves more operators whose orders exceed 2 than operators of order 2, this co-set is composed of all the operators of G which are commutative with less than one half of the operators of H_m . Hence this co-set involves the inverses of all its operators and therefore each of its operators transforms H_m into itself. As one of these operators is of order 2 this co-set and H_m generate a group whose order is twice the order of H_m , and hence the order of each operator of this co-set is a divisor of 4.

* G. A. Miller, *Annals of Mathematics*, vol. 7 (1906), p. 57.

We proceed to prove that whenever g is not of the form 2^a then exactly one half of the operators in each of the co-sets Ht_α , $2 \leq \alpha \leq \gamma$, are of order 2. If this condition were not satisfied, H_m would involve an operator s_1 which would be commutative with an operator t of odd order contained in G and would transform into its inverse an operator t' of order 4 found in the co-set in which more than half the operators would be of order 4. This follows directly from the fact that all the operators of the co-set involving t are commutative with exactly half the operators of H_m while t' is commutative with at most one fourth of the operators of this subgroup.

Let H_1 be the subgroup composed of all the operators of G which are commutative with s_1 . If the product of t' and an operator t_1' of order 2 which is found in H_1 but not in H_m had an order larger than 2 then t_1' and t' would be commutative since this product would be transformed into its inverse by s_1 and hence $(t't_1')^{-1} = t'^{-1}t_1'$. Therefore, it results that t' is transformed either into itself or into its inverse by all the operators of the group generated by the operators of order 2 found in H_1 but not in H_m . These operators clearly generate H_1 , since H_1 involves an operator of odd order and this operator must be contained in each of its subgroups of index 2.

Since t could not transform t' into its inverse it follows that t and t' are commutative. Their product must be transformed into its inverse by s_1 and hence we are led to the contradictory equation

$$(t't)^{-1} = tt'^{-1}.$$

As the assumptions that one of the given co-sets contains more operators of order 4 than of order 2 and that the order of G is divisible by an odd prime number led to a contradiction, we have proved that exactly half of the operators of each of these co-sets must be of order 2 whenever g is divisible by an odd prime.

It will now be proved that the subgroup of index 2 under H_m composed of all the operators of H_m which are commutative with t_α is the same for every value of α from 2 to γ . If this were not true, the subgroup formed by the operators of H_m which are commutative with t would involve an operator s_1 which would transform into its inverse an operator t_1 not found in the subgroup H_1 composed of all the operators of G which are commutative with s_1 . Just as before, we may

prove that t_1 is commutative with t because it is transformed either into itself or into its inverse by all the operators of order 2 contained in H_1 but not in H_m .

Moreover, tt_1 is transformed into $t_1^{-1}t^{-1} = t^{-1}t_1^{-1}$ whenever the order of this product exceeds 2. If this order were 2, s_1 would transform $(tt_1)^2 = t^2t_1^2$ into $t^2t_1^{-2} = t^2t_1^2$. As this leads to a contradiction and as tt_1 is also transformed into tt_1^{-1} by s_1 , it results that the two assumptions that G contains operators of odd order and that some of the operators of H_m which are commutative with a certain t_α are not commutative with every t_α , $2 \leq \alpha \leq \gamma$, are contradictory. It therefore results that *exactly half of the operators of H_m constitute the central of G whenever g is divisible by an odd prime number.*

Let K represent the central of G and suppose that g is divisible by an odd prime number. The quotient group G/K has an order which is divisible by all the odd divisors of g and at least one half of its operators are of order 2. If exactly half of these operators are of order 2, this quotient group is either the dihedral or the generalized dihedral group whose order is of the form $2(2m + 1)$. If more than one half of its operators are of order 2, we may proceed as above and find a second quotient group in which at least one half of the operators are of order 2. Hence we have established the following theorem: *If the order of a group is $2^a(2m + 1)$, $m > 0$, and if more than one half of its operators are of order 2 then this group contains an invariant subgroup of order 2^{a-1} and the corresponding quotient group is either the dihedral or the generalized dihedral group of order $2(2m + 1)$.*

The subgroup of G which corresponds to the subgroup of order $2m + 1$ in the quotient group does not involve any of the operators of H_m which transform each of the operators of G whose orders exceed 2 into their inverses, since more than one half of the operators of the former subgroup have orders greater than 2. This subgroup must be abelian since all of its operators whose orders exceed 2 correspond to their inverses in an automorphism and the products of these operators must also correspond to their inverses. It therefore results from the preceding theorem that, *if a group whose order is divisible by an odd prime number has the property that at least one half of its operators are of order 2, it is either a dihedral or a generalized dihedral group.* It also results that H_m is identical with H_1 whenever g has an odd prime factor but this is not necessarily true when g is of the form 2^a , as can easily be verified.

THE DERIVATIVE OF A FUNCTIONAL.

BY PROFESSOR P. J. DANIELL.

IN his book on Integral Equations* Volterra has given a definition of the derivative of a functional and has stated somewhat restricted conditions under which the variation can be expressed as a linear integral. In the present paper it is shown that, under more general conditions, the variation is a linear functional in the sense of Riesz* and, therefore, a Stieltjes integral. This theorem is assumed as a condition in a paper by Fréchet.† Let

$$F[f(x)]$$

denote a functional F of a continuous function $f(x)$ ($a \leq x \leq b$). With Volterra we shall consider only continuous functions. Let us denote the first variation by

$$D(f; \varphi) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{F[f + \epsilon\varphi] - F[f]\}.$$

In place of Volterra's four conditions we take the two following:

I. $F[f]$ satisfies the Cauchy-Lipschitz condition, namely that we can find a number M such that

$$|F[f_1] - F[f_2]| \leq M \max |f_1(x) - f_2(x)|.$$

II. The first variation $D(f'; \varphi)$ exists for all continuous φ , and all continuous f' in the neighborhood of f ; that is to say that a number $\eta > 0$ can be found so that the variation exists so long as

$$\max |f'(x) - f(x)| \leq \eta.$$

Under these conditions the variation is a linear functional, and therefore a Stieltjes integral,

$$D(f; \varphi) = \int_a^b \varphi(x) d\alpha(x).$$

* V. Volterra, *Equations Intégrales*, p. 12 et seq. F. Riesz, *Annales de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 9.

† M. Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 135.

In the first place

$$(1) \quad \begin{aligned} D(f; l\varphi) &= \lim_{\epsilon \rightarrow 0} \frac{l}{\epsilon} \{F[f + \epsilon l\varphi] - F[f]\} \\ &= lD(f; \varphi). \end{aligned}$$

If we choose $\epsilon > 0$ so small that

$$\epsilon|l|\max|\varphi_1| + \epsilon|m|\max|\varphi_2| < \eta,$$

$D(f + \epsilon m\varphi_2; \varphi_1)$ will exist by II, and

$$F[f + \epsilon l\varphi_1 + \epsilon m\varphi_2] - F[f + \epsilon m\varphi_2] = \epsilon lD(f + \epsilon m\varphi_2; \varphi_1) + \epsilon\delta,$$

$$F[f + \epsilon l\varphi_1] - F[f] = \epsilon lD(f; \varphi_1) + \epsilon\delta',$$

where δ, δ' approach 0 with ϵ . Then

$$(2) \quad \begin{aligned} P(l\varphi_1, m\varphi_2) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{F[f + \epsilon l\varphi_1 + \epsilon m\varphi_2] - F[f + \epsilon m\varphi_2] \\ &\quad - F[f + \epsilon l\varphi_1] + F[f]\} \\ &= l \lim_{\epsilon \rightarrow 0} [D(f + \epsilon m\varphi_2; \varphi_1) - D(f; \varphi_1)]. \end{aligned}$$

Similarly

$$(3) \quad P(l\varphi_1, m\varphi_2) = m \lim_{\epsilon \rightarrow 0} [D(f + \epsilon l\varphi_1; \varphi_2) - D(f; \varphi_2)].$$

The expression (2) is the product of l and a function of m independent of l , while (3) is the product of m and a function of l only, and they are equal. Each must be a product of lm and an expression K independent of l, m .

$$P(l\varphi_1, m\varphi_2) = lmK(\varphi_1, \varphi_2).$$

In this make $l = 1 = m$; then

$$P(\varphi_1, \varphi_2) = K(\varphi_1, \varphi_2).$$

Or

$$P(l\varphi_1, m\varphi_2) = lmP(\varphi_1, \varphi_2).$$

Making $m = l$,

$$P(l\varphi_1, l\varphi_2) = l^2P(\varphi_1, \varphi_2).$$

But

$$\begin{aligned} P(l\varphi_1, l\varphi_2) &= l \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon l} \{F[f + \epsilon l\varphi_1 + \epsilon l\varphi_2] \\ &\quad - F[f + \epsilon l\varphi_2] - F[f + \epsilon l\varphi_1] + F[f]\} \\ &= lP(\varphi_1, \varphi_2). \end{aligned}$$

$$\therefore l^2P(\varphi_1, \varphi_2) = lP(\varphi_1, \varphi_2)$$

$$P(\varphi_1, \varphi_2) = 0.$$

Or

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{F[f + \epsilon\varphi_1 + \epsilon\varphi_2] - F[f] - F[f + \epsilon\varphi_2] + F[f] - F[f + \epsilon\varphi_1] + F[f]\} = 0,$$

$$(4) \quad D(f; \varphi_1 + \varphi_2) - D(f; \varphi_2) - D(f; \varphi_1) = 0.$$

Combining (1) and (4), we see that

$$D(f; c_1\varphi_1 + c_2\varphi_2) = c_1D(f; \varphi_1) + c_2D(f; \varphi_2).$$

Thus the variation is distributive in φ . Secondly, from condition I,

$$|F[f + \epsilon\varphi] - F[f]| \leq M\epsilon \max |\varphi|,$$

or

$$|D(f; \varphi)| = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} |F[f + \epsilon\varphi] - F[f]| \leq M \max |\varphi|.$$

The variation is also bounded, considered as an operation on φ . This proves it to be a linear functional by Riesz's definition. To find the integrating function $\alpha(x)$ we may proceed as follows:

Let $\varphi(x; c, d)$ denote the continuous function,

$$\begin{aligned} \varphi &= 1, & a \leq x \leq c, \\ &= 0, & d \leq x \leq b, \\ &\varphi \text{ linear from } c \text{ to } d. \end{aligned}$$

Then

$$\alpha(c) = \lim_{d \rightarrow c} D(f; \varphi),$$

and in general for any continuous $\varphi(x)$,

$$D(f; \varphi) = \int_a^b \varphi(x) d\alpha(x).$$

RICE INSTITUTE,
HOUSTON, TEXAS.

SCRITTI MATEMATICI.

Scritti matematici offerti ad Enrico D'Ovidio in occasione del suo LXXV genetliaco, 11 Agosto 1918, e pubblicati per cura di FRANCESCO GERBALDI e GINO LORIA. Torino, Fratelli Bocca, editori, 1918. 8vo, 386 pages. Lire 30.

THE title of this volume sufficiently indicates its general character. A beautiful portrait forms the frontispiece, representing Professor D'Ovidio as he will be remembered by many hundreds of Italians as well as by a number of Americans who, like the writer, have had the good fortune, at one time or another, to be among those in attendance at the University of Turin.

There is a short preface written by Professors Gerbaldi and Loria in which is set forth the *raison d'être* for the volume. To quote the opening sentence—"On the approach of the day on which an inflexible law would retire Senator Enrico D'Ovidio from the university chair, there arose in the minds of many students whom he has had in his long and glorious career as a teacher, the pleasant idea of choosing this occasion—which coincides with his 75th birthday—to manifest to him their sentiments of unalterable affection and, at the same time, to present to him their sincere good wishes *ad multos annos*."

Of the 103 persons contributing to the expense of publication, and whose names appear directly following the preface, 47 have been actual students under Professor D'Ovidio, and 20 have been, at one time or another, directly associated with him as *Assistenti* in the University of Turin. The list contains many names well known to students of mathematics the world over, and bears witness to the great influence Professor D'Ovidio has had upon the growth of mathematics and upon the teaching of mathematics, not alone in Italy, but, mainly perhaps through his disciples, throughout the civilized world. This influence has been exerted from the University of Turin for more than forty years.

Following the list of contributors is a catalogue of 91 publications by Professor D'Ovidio. Some of these are problems or solutions of problems; two or three are elementary texts; a number are commemorative addresses or biographical

papers; the majority deal with problems in geometry; several, the most important perhaps, are concerned with the theory of binary forms. A glance at the titles of these papers leads to the conviction that Professor D'Ovidio's great influence has come about rather more through personal contact as a teacher than through published writings, and goes to show that there is such a thing as creative teaching as well as there is creative scholarship—a fact worth noting in a time when so much emphasis is placed upon the latter function and apparently so little is thought about the former.

It would be out of place here to review in detail the twenty-one papers forming the body of the volume. The titles and authors of these papers with perhaps a word or two indicating the content must suffice.

I. "Su alcune classi particolari di sistemi continui di quadriche, e sui rispettivi involuppi." CORRADO SEGRE, Turin.

The lines of ordinary 3-space can be imaged as points upon a V_4^2 in S_5 . A study of point loci upon the V_4^2 leads to properties of ruled surfaces in S_3 . In particular, sections of the V_4^2 by the planes in S_5 are images of quadratic reguli in S_3 . Professor Segre is thus led to what he calls concatenated quadric surfaces and quartic curves of the first kind, and to systems of concatenated quadrics and quartics. A quadric surface and a quartic curve of the first kind lying on the surface are concatenated, or chained together, if a simple quadrilateral inscribed in the latter has its sides lying upon the former. The present paper is evidently an outgrowth of a recent paper by Professor Segre in *Annali di Matematica* (3) 27, (1918), page 151.

II. "Le frazione continue di Halphen." FRANCESCO GERBALDI, Pavia.

The continued fractions of Halphen (*Traité des Fonctions elliptiques*, second volume) arise from the expression $\frac{\sqrt{X} - \sqrt{Y}}{x - y}$, where X and Y are the values of a biquadratic expression when the variable assumes the values x and y , respectively. Halphen, and Jacobi before him, made use of elliptic functions in arriving at the development of irrational expressions like the above in continued fractions. Professor Gerbaldi returns to the more elementary methods of Abel in the present paper.

Two other papers on the same subject have appeared;—one in Volume 53 of *Atti di Torino*, and the other in Volume 51 of *Rendiconti R. I. Lombardo*.

III. “Le cubiche gobbe aventi ciascuno all’infinito tre punti reali e distinti.” GINO LORIA, Genoa.

In 1879 D’Ovidio published an exhaustive study of twisted cubics making use of the symbolic notation of binary forms (*Memorie R. Accademia Torino* and *Giornale di Matematiche*). Professor Loria here studies the cubic hyperbolas, using only elementary means to arrive at the properties of these curves.

IV. “Intorno ad un tipo notevoli di sistemi lineari di reciprocità degeneri tra spazi ad n dimensioni.” EUGENIO TOGLIATTI, Turin.

The study of projective correspondences between hyperspaces leads to the consideration of types of degenerate transformations (cf. Bertini; *Introduzione alla Geometria proiettiva degli Iperspazi*). The present paper follows and is an extension of a note by the same author published in *Atti di Torino*, 1916–17.

V. “Sulle congruenze W di cui una falda focale e una quadrica.” ALESSANDRO TERRACINI, Turin.

One may consider a correspondence set up in which homologous points are the foci on the same ray of a congruence. If, in this correspondence, asymptotic lines on the focal mantels of the congruence are homologous lines, the congruence is called a W -congruence. The author here studies the W -congruences in which one of the focal mantels is a quadric.

VI. “Alcune osservazioni relative ai problemi secondarii della balistica esterna.” GUIDO FUBINI, Turin.

A study of the effect of the wind and of the rotation of the earth upon projectiles, with the hope of modifying the classical differential equations involved so that theoretical results will be in closer accord with experimental data.

VII. “Sulle curve che posseggono una infinità continua di corrispondenze algebriche.” GUIDO CASTELNUOVO, Rome.

A curve of genus greater than unity does not admit of an infinity of birational transformations into itself (Schwarz,

Crelle, 1875). Castelnuovo here studies those curves each of which is transformable into itself by a continuous infinity of non-birational algebraic correspondences and arrives at a generalization of the Schwarz theorem. The paper is based upon the classic memoir of Hurwitz (*Mathematische Annalen*, 1886).

VIII. "Le oscillazione armoniche nelle antenne radiotelegrafiche direttamente eccitate." LUIGI LOMBARDI, Naples.

This paper contains results of some experiments with radio apparatus at the polytechnic school in Naples.

IX. "Sugli integrali semplici di 1^a specie appartenenti ad una superficie algebrica." FRANCESCO SEVERI, Padua.

A development of one of Severi's brief notes communicated to the Académie des Sciences, Paris, in 1911 (*Comptes Rendus*, volume 152, page 1079).

X. "Sopra alcune applicazioni della teoria dell'urto." EMILIO ALMANZI, Rome.

A brief study of the theory of impact in a system of moving bodies with application, in particular, to three spheres.

XI. "Generalizzazione di una trasformazione di d'Ocagne." ANGELO PENSA, Turin.

The transformations of d'Ocagne may be found explained in an article by M. d'Ocagne, *American Journal*, volume 11 (1889), pages 55-70. Pensa here generalizes the problems raised by d'Ocagne (solved in particular cases by himself and others) and reaches general results by means of vector analysis.

XII. "Estensione e studio di un metodo di sommazione generico di Borel." GUSTAVO SANNIA, Cagliari.

This paper follows one by the same author published in *Rendiconti del Circolo Matematico di Palermo*, volume 42, 1917. Both papers are based upon Borel's summation formula (*Leçons sur les Séries divergentes*, Gauthier-Villars, 1901).

XIII. "Sopra la propagazione di onde in un mezzo indefinito." ERNESTO LAURA, Pavia.

This paper deals with the motion of waves in a fluid medium when, in particular, the waves are reflected from a fixed surface. It follows an article by the same author published in *Atti di Torino*, 1915.

XIV. "Problemi sulla determinazione delle linee sghembe."

MATTEO BOTTASSO,* Messina.

This paper deals with certain generalizations of the transformation of Combescure (Bianchi, *Lezione di Geometria Differenziale*, second edition, page 40). It follows a note by Bottasso published in *Atti di Torino*, 1917-18.

XV. "Riflessioni sopra alcuni principii della teoria degli aggregati e della funzioni." BEPPO LEVI, Parma.

A logico-mathematical discussion dealing in particular with the Zermelo postulate (*Mathematische Annalen*, volume 59).

XVI. "Nuovo metodo per la risoluzione diretta dell'equazione $ax + by = c$ in numeri interi e positivi, quando i tre numeri noti a , b , c sono interi e positivi." GIUSEPPE BERNARDI, Bologna.

A short paper whose title sufficiently indicates its content.

XVII. "Un interessante problema di geodesia pratica."

NICODEMO JADANZA, Turin.

A discussion of the problem: To determine the mean error of two sides of a geodetic triangle when the third side and the three angles are measured.

XVIII. "Resto nelle formule di interpolazione." GIUSEPPE PEANO, Turin.

A proof of an interesting theorem in determinants from which results, in particular, the well known Lagrange interpolation formula together with a remainder.

XIX. "Questioni elementari di massimo e minimo." FILIBERTO CASTELLANO, Turin.

Some problems in maxima and minima treated by algebraic equations and inequalities.

XX. "Sulle varietà algebriche a tre dimensioni a superficie-sezioni razionali." GINO FANO, Turin.

An abstract from a memoir with the same title published in *Annali di Matematica*, (3), volume 24 (1915).

XXI. "Introduzione alla teoria della forme in più serie di variabili." GIOVANNI GIAMBELLI, Messina.

A generalization of the theory of connexes (Clebsch-

* Died Oct. 3, 1918.

Lindemann). The paper follows two others on the same subject published in Turin by the author in 1910 and 1912.

One cannot turn aside from reading these papers, however cursorily, without a feeling of profound admiration for the scholars who have made the book possible, and especially at a time when the strain of the war was still in force. Each paper is a distinct contribution to knowledge or else a fuller development of such a contribution recently commenced. The total is a wholly worthy epitome of scientific activity even in normal times. It must, indeed, be a source of great satisfaction to Professor D'Ovidio to have so distinct a proof of the esteem with which his many students, associates, and friends regard his long service and his personal qualifications as an inspiring teacher. One cannot do better, in this connection, than to quote again from the preface. "And we are certain that to the loved teacher our publication will be doubly gratifying in as much as it serves also to show how Italy, in the tragic hours in which we live—not less than in the more grave and decisive periods of her earlier struggles for redemption—has not ceased to feed the sacred flame of science."

L. WAYLAND DOWLING.

SHORTER NOTICES.

Plane Geometry. By E. LONG and W. C. BRENKE. New York, The Century Company, 1916.

THIS text has several very good features. First, before a theorem is demonstrated a method of attack is given. Second, frequent use is made of algebra and thereby many blind proofs are avoided. Third, construction work is introduced early in the course. Fourth, areas are introduced before proportion. Fifth, a little trigonometry and analytics is given.

The main fault with the book is that it contains quite a number of inaccurate statements, e. g., "Place the triangles with their longest sides together," page 32; page 107, $c^2 = h^2 + (b - a)^2$ is only true if A is acute and that is not at all necessary; page 204, the definitions of the trigonometric functions are incorrect.

F. M. MORGAN.

Plane Geometry. By JOHN H. WILLIAMS and KENNETH P. WILLIAMS. Chicago, Lyons and Carnahan, 1915.

THIS text in my estimation is an excellent example of what a plane geometry book should not be. It starts with a lot of formal definitions followed by demonstrative work. There are so many inaccurate and incorrect statements, definitions, and proofs that I think it undesirable to list them. I shall however state a few: "An exterior angle of a triangle is the angle formed by producing one side of the triangle"; "A triangle is defined as a polygon bounded by three straight lines," and twenty pages later we find the definition of a polygon. "The limit of a variable is a constant which the variable is supposed to approach in value and can be made to differ from it by an amount that is less than any assignable value, but can not be made absolutely equal to it."

F. M. MORGAN.

Theories of Energy. By HORACE PERRY. New York, G. P. Putnam's Sons, 1918. vii + 231 pp.

THE title of this book is as misleading as the results are unsatisfactory. The author does not consider "theories of energy," but advances a theory of energy due to his own reading and reflection. The plural character is due to the fact that he considers as theories: theory of energial propagation, theory of the energetic atom, theory of spectral lines, theory of gravity, theory of color, etc. The unsatisfactory character lies in both the results of his reflections and the gaps in his theory.

In the first place (and to endeavor to point out the very long entire list of features of his explanation of energy would be tedious and unprofitable) we need to notice his idea of matter. He begins on the first page with the assertion that "All space is filled with matter, and in the infiniteness of space there is no vacuity anywhere, not even of the extent of an atom's size, and the universe, embracing all the matter in existence, is continuous throughout." This idea that the entirety of space is filled with a continuous material medium was new when Thomson's vortex atom was at its best, but is far in the past at present. Perry's ether is perfectly continuous in all space, but as it has no cohesion between its parts, there is "merely a togetherness without any forcible hold." It is a "perfect fluid with perfect passability."

This ether is subject to condensations and rarefactions, which are propagated in all directions. The character of the wave thus produced seems not to enter his calculations.

Matter is continuous. In fact the electron is not anywhere mentioned in the theory he advances. It is endowed with energy in the form of heat and chemical energy. The radioactive disintegration is merely the chipping at the surface of the atom due to the motion of the internal energy. Energy is then defined to be the internal action of the atom, all energy being *densitic* in character, which means it consists of condensation or rarefaction waves. The wave frequency differentiates the various forms of energy. Gravity on this scheme is of a single frequency.

Magnetism is energy generated in the centers of the iron atoms, being "the natural energy of the iron, modified by the substance with which the iron is combined in molecules, and augmented through a certain method of reciprocal energization." The remaining definitions may be passed over. The author seems to be familiar with a number of antiquated textbooks of a college grade, and somewhat with modern phenomena. From these he has attempted to build up an explanation of the very intricate laws and phenomena of the whole of physics. The result is what would have been expected.

JAMES BYRNIE SHAW.

NOTES.

THE April number (volume 20, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Memoir on the general theory of surfaces and rectilinear congruences," by G. M. GREEN; "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic," by O. E. GLENN; "Concerning a set of postulates for plane analysis situs," by R. L. MOORE; "On the limit functions of sequences of continuous functions converging relatively uniformly," by E. W. CHITTENDEN.

At the meeting of the National Academy of Sciences held at Washington April 28-30, Professors OSWALD VEBLEN, E. J.

WILCZYNSKI, and E. B. WILSON were elected members. Professor EDWARD KASNER read a paper on "Geometry of the wave equation."

At the meeting of the Edinburgh mathematical society on May 9, the following papers were read: By W. H. METZLER, "Theorems concerning the differentiation of a circulant;" by T. M. MACROBERT, "The integrals of the hypergeometric equation."

THE Indian mathematical society held its second conference at Bombay, January 11-13, in connection with the session of the Indian science congress. Professor A. C. L. WILKINSON, of Elphinstone College, Bombay, is president. The society has a membership of 195.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (academic year 1919-1920).—By Professor T. S. FISKE: Differential equations, four hours.—By Professor F. N. COLE: Invariants and higher plane curves, three hours (second term); Theory of groups, three hours.—By Professor JAMES MACLAY: Theory of functions, four hours (first term).—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor C. J. KEYSER: Modern theories in geometry, four hours.—By Professor EDWARD KASNER: Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. J. F. RITT: Transcendental functions, three hours (second term).

CORNELL UNIVERSITY (academic year 1919-1920).—By Professor JAMES MCMAHON: Mathematics of insurance and probabilities; Actuarial science.—By Professor J. H. TANNER: Mathematics of finance.—By Professor VIRGIL SNYDER: Birationals transformations; Theory of equations (second term).—By Professor F. R. SHARPE: Fourier series and the potential function.—By Professor ARTHUR RANUM: Non-euclidean geometry (first term); Theory of numbers (second term).—By Professor W. B. CARVER: Projective geometry.—By Professor D. C. GILLESPIE: Elementary differential equations;

Calculus of variations.—By Professor W. A. HURWITZ: Theory of groups; Vector analysis.—By Professor C. F. CRAIG: Theory of differential equations; Mathematics for teachers.—By Professor F. W. OWENS: Advanced calculus; Mechanics.—By Dr. H. B. OWENS: Advanced analytic geometry.

All courses are three hours a week.

HARVARD UNIVERSITY (academic year 1919–1920).—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Differential and integral calculus (advanced course); Infinite series and products;* Galois's theory of equations.*—By Professor C. L. BOUTON: The elementary theory of differential equations;* Differential equations, with an introduction to Lie's theory of continuous groups.—By Professor J. L. COOLIDGE: Introduction to modern geometry and modern algebra; Projective geometry;* Non-euclidean geometry.*—By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics.*—By Professor O. D. KELLOGG (of the University of Missouri): Introduction to the theory of potential functions and Laplace's equation;* Vector analysis.*—By Professor G. D. BIRKHOFF: Differential and integral calculus (advanced course);* The analytical theory of heat and problems in elastic vibrations;* The partial differential equations of mathematical physics.—By Professor ———: The theory of functions (introductory course); Developments in series;* Dynamics (second course);* By Professor W. C. GRAUSTEIN: Differential geometry of curves and surfaces.—By Dr. H. C. M. MORSE: Elliptic functions;* Automorphic functions.*—By Dr. I. A. BARNETT: Integral equations;* Functions of lines.*

Professor KELLOGG will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor KELLOGG in analysis, by Professor BIRKHOFF in the theory of differential equations, and by Professor GRAUSTEIN in geometry.

PRINCETON UNIVERSITY (academic year 1919–1920).—By Professor H. B. FINE: Functions of a complex variable.—By Professor L. P. EISENHART: Differential geometry.—By

Professor OSWALD VEBLEN: Seminar in analysis situs.—By Professor PIERRE BOUTROUX: Linear differential equations.—By Professor J. H. M. WEDDERBURN: Higher algebra.

All courses are three hours a week.

UNIVERSITY OF CHICAGO. Autumn quarter:—By Professor E. H. MOORE: Seminar in general analysis, I, two hours; Matrices in general analysis, four hours.—By Professor G. A. BLISS: Theory of functions of a real variable, four hours; Calculus, I, five hours.—By Professor L. E. DICKSON: Continuous groups, four hours; Elementary theory of equations, four hours.—By Professor H. E. SLAUGHT: Differential equations, four hours.—By Professor A. C. LUNN: Vector analysis, four hours; Applied mathematics, five hours. Winter quarter:—By Professor E. H. MOORE: Seminar in general analysis, II, two hours; Theory of functions of infinitely many variables in general analysis, four hours.—By Professor G. A. BLISS: Calculus of variations, five hours; Calculus, II, five hours.—By Professor L. E. DICKSON: Theory of algebraic invariants, four hours; Solid analytics, four hours.—By Professor H. E. SLAUGHT: Theory of definite integrals, four hours.—By Professor E. J. WILCZYNSKI: Projective geometry, I, four hours.—By Professor A. C. LUNN: Applications of vector analysis in the theory of electromagnetism, four hours; Applied mathematics, II, five hours.—By Professor J. W. A. YOUNG: Calculus, I, five hours. Spring quarter:—By Professor E. H. MOORE: Seminar in general analysis, III, two hours; Theory of functions of infinitely many variables in general analysis, II, four hours.—By Professor G. A. BLISS: Functions of lines, four hours; Calculus, III, five hours.—By Professor L. E. DICKSON: Finite groups, four hours.—By Professor E. J. WILCZYNSKI: Higher geometry, four hours; Projective geometry, II, four hours.—By Professor A. C. LUNN: The theory of relativity, four hours; Applied mathematics, III, five hours.—By Professor J. W. A. YOUNG: Limits and series, four hours; Calculus, II, five hours.

UNIVERSITY OF ILLINOIS (academic year 1919–1920).—By Professor E. J. TOWNSEND: Differential equations and advanced calculus; Functions of real variables.—By Professor G. A. MILLER: Theory of equations (first term); Theory of groups, II.—By Professor J. B. SHAW: Fundamental functions .

(first term); Functional transformations (second term).—By Professor A. B. COBLE: Automorphic functions; Solid analytic geometry (second term).—By Professor R. D. CARMICHAEL: Linear difference equations.—By Professor ARNOLD EMCH: Algebraic surfaces.—By Professor A. R. CRATHORNE: Actuarial theory.—By Professor A. J. KEMPNER: Modern algebra.—By Dr. E. B. LYTLE: History of mathematics, two hours (second term).—By Dr. HENRY BLUMBERG: Projective geometry. All courses are three hours a week unless otherwise designated.

UNIVERSITY OF PENNSYLVANIA (summer session, July 8 to August 16).—By Professor G. H. HALLETT: Higher calculus, five hours.—By Professor H. H. MITCHELL: Mathematical theory of probability, five hours.—By Professor R. L. MOORE: Introduction to the theory of functions of a complex variable, five hours.

YALE UNIVERSITY (academic year 1919–1920).—By Professor JAMES PIERPONT: Elliptic functions.—By Professor P. F. SMITH: Foundations of geometry.—By Professor E. W. BROWN: Hydrodynamics with applications to aeronautics.—By Professor W. R. LONGLEY: Theory of differential equations.—By Professor J. I. TRACEY: Modern geometry, including differential geometry (a first course).—By Mr. J. K. WHITTEMORE: Differential geometry.—By Dr. J. R. KLINE: Advanced algebra. All courses are two hours a week.

THE announcement in the May BULLETIN of the death of Professor H. G. ZEUTHEN was an error which the responsible editor is more than glad to correct. Professor Zeuthen celebrated on February 15 his eightieth birthday. The BULLETIN wishes this Nestor of Danish mathematicians many happy returns.

AT Harvard University, assistant professor G. D. BIRKHOFF has been promoted to a full professorship of mathematics. Assistant professor W. C. GRAUSTEIN, of Rice Institute, has been appointed assistant professor, and Mr. B. H. BROWN and Mr. C. A. RUPP, JR., have been appointed instructors. Professor O. D. KELLOGG, on leave of absence from the University of Missouri, has been appointed lecturer for the academic year 1919–1920.

IN the department of mathematics of the Naval academy at Annapolis, the following promotions have been announced: to be professor, Mr. PAUL CAPRON; to be associate professors, Mr. W. J. KING and Mr. J. B. EPPES; to be assistant professors, Mr. J. A. BULLARD, Mr. JOHN TYLER, Mr. ARTHUR KIERNAN, Mr. J. N. GALLOWAY, Mr. ALEXANDER DILLINGHAM, and Dr. G. R. CLEMENTS. Dr. G. H. CRESSE, of the University of Michigan, has been appointed instructor.

AT the University of Minnesota, Professor G. N. BAUER remains on leave of absence until January 1, 1920. Assistant professor DUNHAM JACKSON, of Harvard University, has been appointed professor of mathematics, and Major W. L. HART has been appointed assistant professor. Mr. R. W. BRINK has been promoted from an instructorship to an assistant professorship, and granted leave of absence for the academic year 1919-1920, during which time he will be lecturer at the University of Edinburgh. Miss MINNA J. SCHICK has been appointed instructor in mathematics.

DR. C. E. WILDER has been appointed assistant professor of mathematics at Northwestern University.

MR. P. A. FRALEIGH, of Cornell University, has been appointed instructor in mathematics at Dartmouth College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDERSEN (A. F.), BOHR (H.) og MOLLERUP (J.). *Nyere Undersøgelser over Integralregningen*. Kjøbenhavn, 1917. 63 pp.

BOHR (H.). See ANDERSEN (A. F.).

BOREL (E.). *Die Elemente der Mathematik*. Vom Verfasser genehmigte deutsche Ausgabe von P. Stäckel. Band I: Arithmetik und Algebra nebst den Elementen der Differential-Rechnung. 2te Auflage. Leipzig, Teubner, 1919. 8vo. 16 + 404 pp. M. 12.00

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**A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE**

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and third regular meeting of the Society was held in New York City on Saturday, April 26, 1919, extending through the usual morning and afternoon sessions. This being the first eastern meeting since October, the attendance was large, indicating, as it may be hoped, a revival of the conditions preceding the war. The following sixty-seven members were present:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor Susan R. Benedict, Professor G. D. Birkhoff, Professor B. H. Camp, Professor F. N. Cole, Professor Louise D. Cummings, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Dr. J. V. DePorte, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor A. B. Frizell, Mr. T. C. Fry, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor C. C. Grove, Professor C. O. Gunther, Captain Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. J. R. Kline, Dr. K. W. Lamson, President E. O. Lovett, Professor James Maclay, Professor L. C. Mathewson, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. W. Mullins, Dr. F. D. Murnaghan, Mr. L. S. Odell, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Susan M. Rambo, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor E. D. Roe, Jr., Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Professor H. D. Thompson, Mr. H. S. Vandiver, Major Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Dr. C. E. Wilder, Miss Ella C. Williams, Dr. Emily C. Williams, Professor Ruth G. Wood, Professor J. W. Young.

President Morley occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Mr. N. W. Akimoff, Philadelphia, Pa.; Dr. Tobias Dantzig, Columbia

University; Mr. A. C. Maddox, Guthrie, Okla.; Mr. Montford Morrison, Chicago, Ill.; Professor Ganesh Prasad, Central Hindu College, Benares, India; Mr. F. M. Weida, State University of Iowa; Mr. C. L. E. Wolfe, University of California. Two applications for membership in the Society were received.

It was decided to hold the summer meeting of the Society at the University of Michigan, on September 2-4. Professors Beman, Bliss, Karpinski, Osgood, and the Secretary were appointed a committee on arrangements for this meeting. A committee, consisting of Professors Bliss, Birkhoff, and Veblen, was appointed to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

Professors E. W. Brown, L. E. Dickson, and H. S. White were appointed as representatives of the Society in the Division of Physical Sciences of the National Research Council, and President R. S. Woodward and Professors G. D. Birkhoff and W. D. MacMillan as representatives of the Society in the American Section of the International Astronomical Union.

Professor Archibald, chairman of the committee on the publication of a mathematical year book, presented a preliminary report; the committee was continued and asked to make a further report at a future meeting.

Professor A. B. Coble was appointed to succeed Professor D. R. Curtiss as a member of the Editorial Committee of the *Transactions*.

A special feature of the meeting was the reports on the work in ballistics at Aberdeen and Washington, which occupied the first part of the afternoon session. Titles and abstracts of these reports are included in the list below (papers 14, 15, 16).

About fifty members and friends gathered at the midday luncheon. Thirty-two attended the dinner in the evening. Much satisfaction was expressed at the return of these pleasant occasions.

The following papers were read at this meeting:

(1) Professor C. J. KEYSER: "Concerning groups of dyadic relations in an arbitrary field."

(2) Mr. J. K. WHITEMORE: "Certain functional equations connected with minimal surfaces."

(3) Professor W. B. FITE: "Linear functional differential equations."

(4) Mr. L. B. ROBINSON: "Note on a theorem due to Wilczynski."

(5) Mr. L. B. ROBINSON: "A curious system of polynomials, continued."

(6) Professor O. E. GLENN: "Covariants of binary modular groups."

(7) Professor O. E. GLENN: "Modular covariant theory of the binary quartic. Tables" (preliminary report).

(8) Professor O. E. GLENN: "Invariants of velocity and of acceleration."

(9) Professor F. H. SAFFORD: "Reduction of the elliptic element to the Weierstrass form."

(10) Mr. PHILIP FRANKLIN: "Computation of the complex roots of the function $P(z)$."

(11) Dr. A. R. SCHWEITZER: "On the history of functional equations" (preliminary report).

(12) Professor E. D. ROE, JR.: "The irreducible factors of $1 + x + \dots + x^{n-1}$. Second paper."

(13) Professor E. D. ROE, JR.: "The irreducible factors of a circulant."

(14) Captain DUNHAM JACKSON: "Contributions of modern mathematics to exterior ballistics."

(15) Dr. T. H. GRONWALL: "Qualitative properties of the ballistic trajectory."

(16) Major OSWALD VEBLEN: "Progress in design of artillery projectiles."

(17) Professor G. D. BIRKHOFF: "Boundary value and expansion problem for differential systems of the first order."

(18) Professor G. D. BIRKHOFF: "Note on the closed curves described by a particle moving on a surface in a gravitational field."

(19) Professor G. D. BIRKHOFF: "Note on the problem of three bodies."

(20) Professor EDWARD KASNER: "A characteristic property of central forces."

(21) Dr. J. F. RITT: "On weighting-factor curves for low elevations."

(22) Professor A. C. LUNN: "Some functional equations in the theory of relativity."

(23) Dr. J. R. KLINE: "Concerning sense on closed curves in non-metrical plane analysis situs."

(24) Professor R. L. MOORE: "On the most general class

L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true."

(25) Mr. H. S. VANDIVER: "On the class number of the field $\Omega(e^{2\pi i/p^n})$ and the second case of Fermat's last theorem."

(26) Professor F. W. BEAL: "On certain points of congruences of circles."

(27) Professor L. L. SILVERMAN: "Regular transformations of divergent series and integrals."

(28) Mr. T. C. FRY: "The application of the modern theories of integration to the solution of differential equations."

(29) Dr. C. A. FISCHER: "Completely continuous transformations and Stieltjes integral equations."

(30) Professor ARNOLD EMCH: "On closed curves described by a spherical pendulum."

(31) Professor H. S. WHITE: "An explicit formula for two old problems."

(32) Professor L. P. EISENHART: "Triply conjugate systems with equal point invariants."

Mr. Franklin was introduced by Dr. Gronwall. The papers of Mr. Robinson, Professor Safford, Dr. Schweitzer, Professor Lunn, Professor Beal, Professor Silverman, Mr. Fry, Dr. Fischer, Professor Emch, Professor White, and Professor Eisenhart were read by title. Abstracts of the papers follow below.

1. If the domain of a relation be identical with its co-domain and therefore with its field, then, whatever field be given, any relation of the kind in question belongs to one and but one of the following nine types of relations: one-one, one-some, some-one, some-some, one-many, many-one, some-many, many-some, many-many. These classes together with those formed from them by logical addition constitute a certain class C of classes of relations. Professor Keyser has determined what members of C are groups, the rule of combination being relative multiplication.

2. In Mr. Whittemore's paper functional equations for the Enneper-Weierstrass function defining a real minimal surface, which express certain properties of the surface, are solved, and from the results several theorems are obtained.

3. Professor Fite establishes the existence of solutions of a

certain class of linear functional differential equations and discusses the properties of the functions defined by certain mixed difference equations.

4. At a previous meeting of the Society (April 28, 1917) Mr. Robinson showed that the general system of linear homogeneous differential equations could be reduced to a canonical form whose covariants could be computed by a finite number of quadratures. This result was accepted for publication by the editors of the *Johns Hopkins Circular*. Subsequently the author discovered that his method yields a very neat proof of the theorem enunciated by Wilczynski on page 39 of his *Projective Differential Geometry*.

5. In this paper Mr. Robinson continues a note read at the summer meeting of 1918. Given a system of equations

$$\sum_{j=1}^{n+1} P_{jj} x_{jr}^2 + 2 \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} P_{ki} x_{kr} x_{ir} = 0$$

$$(r = 1, 2, \dots, n), \quad (i > k);$$

polarize each of the above equations with respect to the r following points: $x_{1r}, x_{2r}, \dots, x_{n+1r}$ ($r = 1, 2, \dots, n$). Denote the resulting system by J_2 . The system composed of J_1 and J_2 can be designated by P . Solve J_1 with respect to x_{1r} , substitute the solutions in J_2 , and rationalize. Write the coefficients of the resulting forms thus:

$$C_\lambda(P_{11} \dots P_{n+1, n+1}).$$

The necessary and sufficient condition for the existence of values of the x 's annulling P and not annulling all the determinants of the matrix $\|x_{ij}\|$ ($i = 1, 2, \dots, n+1$) ($j = 1, 2, \dots, n$) is given by the vanishing of all the C 's.

6. Contributions relating to processes, general and enumerative, for complete systems of covariants of binary forms transformed modulo 2, or 3, are treated in Professor Glenn's paper, which is a summary of the tabulated results, and methods, of a number of papers by the author on formal modular covariant theory (mod p).^{*} An algorism of modular

^{*} For a treatment of universal covariants of modular groups, see Dickson, *Transactions*, vol. 12 (1911), p. 75.

convolution, involving simultaneous invariants and covariants, is here given as a new conclusion, and the forms from the complete systems of former papers are exhibited under the notation of this algorism.

The paper appeared in the April number of the *Proceedings of the National Academy of Sciences*.

7. The doctrine of the covariant scale and of the construction of a complete formal modular concomitant system by a process of passing from degree to degree, developed by Professor Glenn in former papers, now enables him to derive fundamental systems of the binary quartic form with reference to the linear congruence groups $G_6 \pmod{2}$, and $G_{48} \pmod{3}$. He also extends the general theory of the forms of even order transformed modulus 2 and 3, and summarizes certain conclusions in this theory in covariant tables for the forms of orders 4, 6, 8. The problem of the complete seminvariant system is, in the methods of this paper, a phase of a theorem on the reducibility modulo p of the general quartic of order $> p^2 - 1$.

8. In relativity theory the velocity $(\dot{x}, \dot{y}, \dot{z})$ of a point moving uniformly, as observed from a system of reference S , is connected with the apparent velocity $(\dot{x}', \dot{y}', \dot{z}')$ for an observer situated upon a system S' , moving relatively to S , by a set of equations of substitution upon the variables $\dot{x}, \dot{y}, \dot{z}; \dot{x}', \dot{y}', \dot{z}'$. Professor Glenn constructs systems of concomitants, both universal and quantitative, in each of two domains, for these transformations. A similar invariant theory of acceleration is developed.

9. A formula for the reduction of the elliptic element to the Weierstrass form and quoted by Biermann as derived from Weierstrass' lectures has been discussed briefly by Greenhill, Enneper and others. Haentzschel has referred to it as the most general solution of the differential equation satisfied by all elliptic functions, and then used it as a basis of extension and criticism of Wangerin's surfaces in a potential problem. Professor Safford has published several articles concerning Wangerin's and Haentzschel's results, and in this paper gives new facts about the formula, showing in particular that it is not a more general form but arises solely from a change of constants.

10. By a method based on an existence proof recently given by T. H. Gronwall, the numerical values of the four complex roots of the P -function associated with the gamma function have been calculated by Mr. Franklin. The values found are: $-1.7262976 \pm 1.2380921i$ and $-3.7264730 \pm .5406746i$.

11. Dr. Schweitzer compiles a first section of a bibliography (with critical notes) of functional equations, excluding references primarily to difference, differential and integral equations, or mixed types of such equations. The present work, which the author hopes to amplify, is divided into I. Classification; II. Range of application; III. Methods of solution. Subdivisions under I are I_1 . Simple, and I_2 . Composite genesis; I_3 . Analytic, and I_4 . Non-analytic reference, and under I_4 the further divisions I_{41} . General theories of composition, with the sections I_{411} . Equations in partial functions, and I_{412} . Equations in iterative compositions. The operations considered are mainly distribution, inversion, elimination, iteration, transformation as applied to variables or functions. Attention is called to the relations given by Hankel* and Stolz-Gmeiner† when interpreted as functional equations (cf. Hankel: "Calculus of Operations") and their generalization, and the interesting but apparently little known functional equations discussed by E. Schroeder.‡ In the *Mathematische Annalen*, volume 12, page 483, Schroeder gives an attempted, but erroneous, solution of a distributive equation in the domain of substitution groups.

12. In this paper, which is a continuation of a previous one, Professor Roe derives relations among the roots of $1 - x^n = 0$, various formulas for factorization, and the irreducible factors of $f(x, n) \equiv 1 + x + \dots + x^{n-1} \equiv 1, n$, of $f(x^m, n) \equiv m, n$ and of $X_n(x^m) \equiv \frac{m}{n}$. $X_n(x) = 0$ gives the primitive n th roots of unity, and $X_n(x) \equiv \frac{1}{n}$ is irreducible. The paper establishes that an infinite number of identities each of two types comprises all the interfunctional expressibility relations among the f 's and X 's, namely:

$$S_i \equiv F(T_1, T_2, \dots, T_e),$$

* *Theorie des complexen Zahlensysteme* (1867), pp. 26-28, 34, 13, 63, 106 and elsewhere.

† *Theoretische Arithmetik* (1902), chapter III, especially, pp. 52, 53, 38, note 1.

‡ For example, *Archiv* (2) vol. 5; *Crelle*, vol. 90; *Math. Annalen*, vols. 10, 29; and the *Vorles. u. d. Algebra der Logik*, vol. 1 (1890), pp. 617, 630 et seq.

which states that every S can be expressed as an explicit rational function of the T 's, and

$$F_1(S_1, S_2, \dots, S_r) \equiv F_2(T_1, T_2, \dots, T_s).$$

In both identities S 's and T 's represent both of them X 's, or both f 's, or either one f 's and the other X 's, making eight subtypes in all. Their development and discussion form a large part of the paper, and they are used in solving the main problem.

The most general solutions are:

For irreducible factors of the X 's

$$\frac{n_1^{a_1} \dots n_r^{a_r} m_1^{c_1} \dots m_j^{c_j}}{m_1^{b_1} \dots m_k^{b_k}} \equiv \prod_{\sum \lambda_i = \sum a_i} n_1^{\lambda_1} \dots n_r^{\lambda_r} m_1^{\nu_1} \dots m_k^{\nu_k},$$

$\nu_i = b_i + c_i$, $k \geq j \geq 0$, $a_i \geq \lambda_i \geq 0$, and the product contains $\prod_1^r (a_i + 1)$ irreducible factors.

For irreducible factors of the f 's

$$\frac{n_1^{a_1} \dots n_r^{a_r} m_1^{c_1} \dots m_j^{c_j}, m_1^{b_1} \dots m_k^{b_k}}{\prod_{\substack{\sum \lambda_i = \sum a_i, \sum \nu_i = \sum (b_i + c_i) \\ \sum \lambda_i = 0, \sum \nu_i = 1 + \sum c_i}} n_1^{\lambda_1} \dots n_r^{\lambda_r} m_1^{\nu_1} \dots m_k^{\nu_k}},$$

$k \geq j \geq 0$, $c_{j+l} = 0$ when $l > 0$, $a_i \geq \lambda_i \geq 0$, $b_i + c_i \geq \nu_i \geq c_i$, a product that contains

$$\prod_1^r (a_i + 1) \left\{ \prod_1^j (b_i + c_i + 1) \prod_{j+1}^k (b_i + 1) - \prod_1^j (c_i + 1) \right\}$$

irreducible factors.

Numerous applications of the theory and methods are given. It is pointed out that a theorem of Lombardi is contained as a very special case in the second of the preceding general solutions, which is the generalization of all problems of the type of Lombardi's. A theorem proved by De Virieu is also generalized: The necessary and sufficient conditions that $f(x) \equiv \Pi_1^k(x^{\alpha_i} - 1)/\Pi_1^j(x^{\beta_i} - 1)$ is integral are: no prime number enters more β 's than α 's, and its distribution in the individual β 's is not higher than in the individual α 's. If $r = k = p$, $\alpha_i = m - i + 1$, $\beta_i = i$, and x is replaced by x^b , we get the theorem proved by De Virieu. The paper concludes with lists of irreducible factors and applications of the developed theory.

13. The observations contained in this paper will be incorporated in Professor Roe's paper preceding.

14. Captain Jackson's paper gives a brief summary of recent progress in the treatment of the fundamental problems of exterior ballistics under the guidance of modern methods in mathematical analysis. Particular emphasis is placed on Moulton's development of the method of numerical integration for computing trajectories and differential variations, and on Bliss's method of computing differential variations by the use of an adjoint system of differential equations. It is pointed out further that, while the theory of the motion of a projectile, regarded as a material particle, has reached a relatively satisfactory state, present knowledge of the accompanying gyroscopic and fluid motions is highly incomplete.

15. In this paper Dr. Gronwall considers the differential equations for the motion of a heavy particle in a resisting medium $x'' = -Fx'$, $y'' = -Fy' - g$, where F is the resistance divided by the velocity. A qualitative study of the trajectories belonging to this system is made under the sole assumption that $F = F(x', y', x, y, t)$ is a *positive* continuous function of its five arguments. Various inequalities involving range, maximum ordinate, time of flight, etc., are derived, all of these inequalities reducing to equalities in vacuum. A summary of further results obtained under stronger assumptions on F is given; these results are of practical importance in the question of deducing simple approximate formulas for the differential corrections in ballistics.

16. The paper by Major Veblen is a report on experimental work carried out at the Aberdeen Proving Ground, which led to improvements in artillery projectiles. Among the questions investigated were the effects of false ogives, of boat-tailed or stream-lined bases, and of devices for modifying the yaw. Considerable increases in range were obtained by these methods, but the most important gain was obtained by a modification of the rotating band by the removal of some excess copper. This resulted, in the case of the 6-inch gun, in an increase in range of about $2\frac{1}{2}$ miles, and in a diminution of dispersion by a factor of 8. Similar results were obtained in several other guns. It is intended to publish an account

of these studies on rotating bands in an article by Major Veblen and Lieutenant Alger in the *Journal of the United States Artillery*.

These investigations were of an empirical nature, and were carried out, for the most part, without the guidance of a mathematical theory. The absence of a mathematical theory is particularly to be underlined in the case of the rotating band experiments, because semi-official popular accounts of this work have been published in which it is stated that the results were obtained by pure mathematics. The work at the proving ground was done in close collaboration by several men, among whom may be mentioned Lieutenants P. L. Alger, F. E. Fish, and H. Galajikian, and Captain J. W. Alexander. Although the coöperation among all concerned was so close as to make it impossible to trace ideas to their sources, the largest share of the credit must be assigned to Lieutenant Alger, particularly in the rotating band investigations. The results on the 6-inch gun were obtained independently in France by Captain R. H. Kent, after studying the Aberdeen results for the 8-inch howitzer.

17. Professor Birkhoff shows that his earlier work on boundary value and expansion problems for a single ordinary linear differential equation of the n th order (*Transactions*, volume 9 (1908), pages 219–231, 373–395) can be extended to differential systems of the first order. In this manner it is possible to deal with these problems for ordinary differential equations of the n th order in which the parameter does not enter linearly.

18. In a paper by Professor Emch (*Proceedings of the National Academy of Sciences*, volume 4 (1918), pages 218–221) the existence of certain types of closed curves of motion for a particle moving on a sphere in a gravitational field is established. Professor Birkhoff points out in his note that if a point moves on any surface which has a vertical plane of symmetry toward which it is concave, then the existence of such closed curves is an immediate consequence of general results deduced by him earlier (*Transactions*, volume 18 (1917), pages 193–300).

19. Using the methods of Sundman (*Acta Mathematica*,

volume 36 (1912), pages 105–195), Professor Birkhoff proves that if three bodies lie sufficiently near together at $t = 0$ with *assigned* area integral constants not all zero, then at least two of the mutual distances become infinite for $\lim t = \infty$.

20. Professor Kasner shows that the differential equation of the third order defining all the trajectories of a plane field of force admits a first integral of the form $y'' = cF(x, y, y')$ when and only when the field is central (or, as a limiting case, parallel). Generalizations are obtained by imposing some instead of all five of the geometric properties characterizing a dynamical family. (See Princeton Colloquium Lectures, page 10.)

21. In this paper Dr. Ritt discusses the manner in which weighting-factor curves for differential variations behave as the elevation approaches zero. It is found that there exists a limiting weighting-factor curve for each type of atmospheric disturbance. In the case of a range wind the weighting-factor curve is a cubic which depends on the muzzle velocity, but not on the projectile. The weighting-factor curves for a variation in air density and for a variation in the velocity of sound have the same weighting-factor curve for zero elevation, a cubic which is the same for all projectiles and for all velocities. The weighting-factor curves for cross wind approach a parabola which is also independent of the projectile and of the velocity. These limiting curves may be used in the reduction of firings at low angles, when it is desired to avoid short-arc computations. This paper will be offered for publication to the *Journal of the U. S. Artillery*.

22. In transcribing his process of light-signalling Einstein obtained relations connecting the space-time coordinates in two systems, in the form of functional equations. He solved these by passing to partial differential equations and restricted the solutions to linearity by appeal to a homogeneity whose meaning is somewhat obscure. In the present paper, through a somewhat more complete use of functional equations, Professor Lunn shows that the assumption of differentiability is not necessary, and that the needed features of homogeneity are already implicit in the optical and kinematic postulates.

23. In a paper, recently published in the *Annals*,* Dr. Kline gave a non-intuitional definition Σ of sense on closed curves in any space satisfying Professor R. L. Moore's system of axioms Σ_3 . It will be remembered that Σ_3 , besides being satisfied by an ordinary euclidean plane, is also satisfied by certain spaces which are neither metrical, descriptive, nor separable. In the present paper, Dr. Kline considers the following set of independent postulates for sense on closed curves with respect to their exterior:

Axiom 1. If A and C separate B and D on the closed curve J , then the sense ABC on J is not the same as the sense ADC on J .

Axiom 2. If, on the arc ABC of the closed curve J , E , H and F are points such that on the arc ABC the order $AEHFC$ holds, while EXF is an arc lying except for its endpoints within J , then the sense ABC on J is the same as the sense EHF on $EHFXE$.

Axiom 3. If the sense $A_1B_1C_1$ on J_1 is both the same as the sense $A_2B_2C_2$ on J_2 and as the sense $A_3B_3C_3$ on J_3 , then the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3 .

The result obtained is that, if Σ' is any definition of sense on closed curves in a space satisfying Σ_3 , then a necessary and sufficient condition that Σ' have the properties of Axioms 1 to 3 is that Σ' be equivalent to Σ .

24. In a class L of Fréchet, a set of points M is said to possess the Heine-Borel-Lebesgue property if, for every infinite family G of subsets of L that covers M , there exists a finite sub-family of G that also covers M . The Heine-Borel-Lebesgue theorem is said to hold true in a given space L if every closed and compact subset of L possesses the Heine-Borel-Lebesgue property. In a recent paper, Fréchet has shown that in order that the Heine-Borel-Lebesgue theorem should hold true in a given class S (i.e., a class L in which the derived set of every set is closed) it is sufficient that that class S should be a class V (a class L in which a distance function exists). He points out however that this sufficient condition is not necessary and raises the question as to what property it is necessary and sufficient that a class S should possess in order that the Heine-Borel-Lebesgue theorem should hold

* Cf. "A definition of sense on closed curves in non-metrical plane analysis situs," *Annals of Mathematics*, 2d series, vol. 19 (1918), pp. 185-200.

true in that class. In the present paper, Professor Moore exhibits one such property. He calls a family G of point sets a monotonic family if, of every two members of G , one of them contains the other one. A monotonic family G is said to be proper if there is no point common to all the members of G . A set of points M is said to have the property K if every monotonic family of closed subsets of M is improper. A necessary and sufficient condition that the Heine-Borel-Lebesgue theorem should hold true in a given class S is that in that class every closed compact point set should have the property K .

If a set of points M is defined as being compact in the new sense in case for every proper monotonic family G of subsets of M there exists at least one point which is a limiting point of every member of G , then the following theorem holds true:

In a class S , in order that a point set M should possess the Heine-Borel-Lebesgue property it is necessary and sufficient that M should be closed and compact in the new sense.

25. In the *Göttinger Nachrichten*, 1910, pages 507–516, F. Bernstein proves that if the class number of the field defined by $e^{2\pi i/p^2}$, where p is prime, is divisible by p but not by p^2 , then the relation

$$(px)^p + y^p + z^p = 0$$

is not satisfied in rational integers x, y, z , none zero. In the present paper Mr. Vandiver investigates the divisibility of $\Omega(e^{2\pi i/p^2})$ by p , and shows that Bernstein's assumption is equivalent to the statement that p is a regular prime, and therefore his result is included in Kummer's well known criterion.

26. Professor Beal's paper is in abstract as follows: With one exception, at most, four points of a circle of a congruence generate surfaces normal to lines through them and in planes orthogonal to the circle. If the circle touches the envelope S of its plane, one, two, or three points may coincide at the point of tangency without imposing a condition on the radius R . If S is developable or if the circle is tangent to an asymptotic line, two coincide.

When the center is at the point of tangency these cases arise: With radius $R(u)$ and lines of curvature parametric,

two points are on the tangent to $v = \text{const.}$ and the normals at them intersect the normal of S at distances $\pm R'(u)\delta^{-1/2}$ from the center of normal curvature of $v = \text{const.}$ * The other points are symmetrical with respect to the tangent of $v = \text{const.}$ and the normals pass through the center of normal curvature of $u = \text{const.}$ The last two points are real only when GR'^2 is less than or equal to $\delta G + GE + 2DD''$. In the first case the points are distinct and in the second coincide on the tangent to $v = \text{const.}$ The above holds true if R is constant. The four points are then on the tangents to the lines of curvature. If S is developable with rectilinear generators $v = \text{const.}$ and orthogonal trajectories $u = \text{const.}$ three cases exist: $(\partial R/\partial u)^2 < E$, four distinct points; $(\partial R/\partial u)^2 = E$, three coincident points; $(\partial R/\partial u)^2 > E$, two imaginary points.

27. Hurwitz and Silverman have studied transformations of sequences of the type $t_n = \sum_{k=1}^n a_{nk} s_k$, where the numbers a_{nk} are defined in terms of certain values of a function $f(z)$ of the complex variable z . Professor Silverman defines a_{nk} by means of a real function $f(x)$ of the real variable x . The condition for regularity of the transformation T_f is obtained simply in terms of $f(x)$ and its continuous derivative $f'(x)$. The condition for the equivalence of two transformations T_f and T_g is that there exist a solution, having certain properties, of an integral equation involving $f(x)$ and $g(x)$. As a special case, the equivalence of Cesàro and Hölder summability of like order, integral or fractional, is deduced. It is further shown that this method in terms of real variables is more general than the one in terms of complex variables.

The results obtained for divergent series are shown to apply also to divergent integrals.

28. The primary object of Mr. Fry's paper is to establish the validity of certain methods of treating differential equations by the use of Fourier's integrals. These methods have been used by a number of mathematicians in the treatment of circuit problems, but the very considerable extent of their field of validity apparently has not been recognized. From

* E, G, D, D'', δ and G are fundamental quantities for S and its spherical representation.

a standpoint of pure mathematics, the most important contribution of the paper is a definition of the integral

$$\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} dn f(\lambda) \phi(n, t) e^{in(t-\lambda)},$$

which applies even when ϕ does not vanish at ∞ . This definition is shown to be consistent with the ordinary Riemann definition, when the latter applies; and the validity of performing addition, subtraction, differentiation and integration under the sign of integration is established. The conception of Stieltjes' integrals is used in carrying out the argument. A minor point of interest concerning the possibility of varying the path of a Fourier integral is touched upon, leading to the reduction of such integrals to closed paths.

29. Riesz has recently derived a large part of the Fredholm theory of integral equations for the equation

$$f(x) = \varphi(x) + \lambda T(\varphi),$$

where $T(\varphi)$ is any completely continuous linear transformation, and in an earlier paper he has proved that every linear transformation, that is linear functional depending on a parameter, can be put in the form

$$T(\varphi) = \int_a^b \varphi(y) d_y K(x, y).$$

After deriving a sufficient condition that a set of discontinuous functions shall be compact, Dr. Fischer has proved that this transformation will be completely continuous if the variation of K in y and its two-dimensional variation are both finite. Another sufficient condition and a necessary condition are also obtained.

30. Professor Emch presents the results of an investigation of the curves described by a spherical pendulum in case that they are closed. The discussion is based upon the parametric representation of the cartesian coordinates of a point of the curve by Weierstrassian σ -functions. It is shown that the closed curves are algebraic, and rational in Greenhill's case, in which the pendulum-bob just reaches the plane of suspension. A classification of these curves is given as well as an

account of their principal geometric properties. The paper was published in the March number of the *Tôhoku Mathematical Journal*.

31. In the theory of conics it is a well known theorem that two inscribed triangles determine a second conic, for which each is self-polar. On a twisted cubic curve a similar theorem was proved by von Staudt and Hurwitz for two tetrahedra. Professor White's paper presents an explicit algebraic formula by which both these theorems are established, and from whose obvious extension others can be inferred. This formula effects in the one case a (2, 2) transformation, in the other a (3, 3) transformation, of the parameter in terms of which points on the locus are rationally expressible. It permits the explicit determination of the infinitely many self-polar triangles of a second conic which are inscribed in the first, as soon as two are known.

32. When the coordinates of a point in space are expressed in terms of three independent parameters u, v, w , the points for which any one of the parameters is constant, say w , lie on a surface. If the parameters are such that the curves $u = \text{const.}$ and $v = \text{const.}$ on the surface form a conjugate system, we say that space is referred to a triply conjugate system of surfaces. Professor Eisenhart proposes and solves the problem of finding triply conjugate systems for which the three Laplace equations satisfied by the cartesian coordinates have equal invariants. When in particular the triple system of surfaces is orthogonal, the lines of curvature on all the surfaces are isothermic. The problem of finding these orthogonal systems has been solved by Darboux. This paper will appear in the June *Annals*.

F. N. COLE,
Secretary.

REPORT ON THE THEORY OF THE GEOMETRY OF NUMBERS.*

BY PROFESSOR H. F. BLICHFELDT.

(Read at the Chicago Symposium of the American Mathematical Society,
March 28, 1919.)

1. *Introduction.*—There arises in theory of numbers an important class of problems of a kind best illustrated by an example: Consider the quadratic form $f = ax^2 + 2bxy + cy^2$, $ac - b^2 > 0$. Let the numerical value of the determinant $D = ac - b^2$ be given, but not the coefficients a, b, c individually; also let it be specified that the variables x, y must be integers (positive or negative or zero). Under these conditions, what can be predicted as to the least absolute value of f , other than $f = 0$? Designating this value by $[f]$, we have† $[f] \leq 2/\sqrt{3D}^{\frac{1}{2}}$, the extreme limit being reached by the form $x^2 + xy + y^2$.

Isolated problems of like nature were studied by prominent mathematicians during the past century. Hermite discovered a superior limit to the least value $[f]$ of a positive definite quadratic form in n variables in terms of n and the numerical value of the determinant D : $[f] \leq (\frac{1}{2})^{(n-1)/2} D^{1/n}$ (*Journal für Mathematik*, volume 40, 1850, page 263); this was the first important result of a general nature.

In the matter of references we shall use the abbreviations: M_1, M_2, M_3, M_4 designate respectively the following books by Minkowski: "Diophantische Approximationen," Leipzig, 1907; "Geometrie der Zahlen," Leipzig, 1896–1910; "Gesammelte Abhandlungen," volumes 1 and 2, Leipzig, 1911.

B_1 refers to "A new principle in the geometry of numbers" by the author, *Transactions American Mathematical Society*, 1914, pages 227–235; B_2 to a paper read before this Society, San Francisco Section, April 6, 1918 (see this BULLETIN, 1918, page 418).

* An exposition of the theory of the geometry of numbers is to appear in the *Annals of Mathematics* during the fall and winter of the present year. For this reason the report here given of the lecture at the Chicago Symposium is very brief. Proofs have been omitted, and only a few illustrative examples are included. The fundamental theorems are, however, stated practically in full (§3).

† See B_1 , p. 233, for references.

2. *Minkowski's "nowhere concave" surfaces* (M_2 , pages 1-76).—It remained for Minkowski to discover a theorem bearing on the least values of a very general class of functions, by means of an elegant geometrical interpretation of this minimum. He defines a real function φ of n real finite variables $x_1, \dots, x_n$ having the following properties (A), (B), (C):

- (A) $\varphi(x_1, \dots, x_n) =$ a definite positive number unless
 $x_1 = 0, \dots, x_n = 0;$
 $\varphi(0, \dots, 0) = 0;$
 $\varphi(tx_1, \dots, tx_n) = t\varphi(x_1, \dots, x_n)$ for any positive number $t;$
- (B) $\varphi(x_1 + y_1, \dots, x_n + y_n) \leq \varphi(x_1, \dots, x_n) + \varphi(y_1, \dots, y_n);$
- (C) $\varphi(-x_1, \dots, -x_n) = \varphi(x_1, \dots, x_n).$

The following functions (1) and (2) may serve as illustrations. We have taken $n = 2$, $x_1 = x$, $x_2 = y$.

$$(1) \quad \varphi = \sqrt{ax^2 + 2bxy + cy^2}, \quad ac - b^2 > 0;$$

$$(2) \quad \varphi = |x| \text{ when } |x| \geq |y|; \quad \varphi = |y| \text{ when } |x| \leq |y|.$$

The corresponding curves $\varphi = k$ are respectively an ellipse and a square.

It may be proved that

- (a) φ is continuous in the variables $x_1, \dots, x_n;$
 (b) the variables satisfying the inequality $\varphi \leq k$, for a given finite positive k , do not exceed in absolute value a certain finite number depending on k and $\varphi;$
 (c) the surface $\varphi = k$ has the origin $(0, \dots, 0)$ as a center of symmetry and is nowhere convex as seen from this center;
 (d) for a given positive k , the points $(x_1, \dots, x_n)$ are separated into three distinct sets: inner points, points on the surface, outer points, according as the following conditions are satisfied: $\varphi < k$, $\varphi = k$, $\varphi > k;$
 (e) the outer and inner volumes of $\varphi = k$ have one and the same value, namely the integral $\int dx_1 dx_2 \dots dx_n$ extended to all the inner points.

It follows that only a finite number of lattice points, (i. e., points whose coordinates are integers (positive or negative or zero), will be inner points or points on the surface $\varphi = k$,

and that the surface will lie in the interior of a cube (n -dimensional) of finite edge. If k is taken small enough, all lattice points other than the origin will be outer points.

3. *General theorems.* For a surface as thus defined, Minkowski proved the following theorems I, I', III:

THEOREM I (M_2 , page 76). *If the volume of $\varphi = k$ is $\geq 2^n$, then at least one lattice point other than $(0, \dots, 0)$ will be an inner point or a point on the surface.*

THEOREM I' (M_2 , page 76). *If the volume of $\varphi = 1$ is represented by J , then there will be at least one set of integers $l_1, \dots, l_n$ not all zero, such that*

$$0 < \varphi(l_1, \dots, l_n) \leq 2/J^{1/n}.$$

The somewhat more general geometrical theorem proved by the author (B_1 , page 228) may be stated as follows in condensed form:

THEOREM II. *A simple closed surface of volume V can be placed in such a position by means of a translation $x_1 = x_1' + a_1, \dots, x_n = x_n' + a_n$ that the number of lattice points which will lie inside or on the surface is $> V$.*

This surface need not possess a center, nor does it need to be concave towards an inner point.

THEOREM III (M_3 , page 270). *There exists a function φ' obtained from φ by means of a linear homogeneous transformation of determinant unity of the variables $x_1, \dots, x_n$, such that*

$$[\varphi'] \geq \left\{ \frac{2 \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots \right)}{J} \right\}^{1/n}.$$

Geometrically: if the volume of $\varphi' = k$ is

$$< 2 \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots \right),$$

then will every lattice point except $(0, \dots, 0)$ be an outer point. The author has obtained slightly better results (B_2).

4. *Applications to homogeneous forms.*

THEOREM IV (M_1 , pages 68–79). *Let $\xi = \lambda x + \lambda' y + \lambda'' z$, $\eta = \mu x + \dots$, $\zeta = \nu x + \dots$ be three linear forms in x, y, z having real coefficients $\lambda, \dots, \nu''$ and a determinant $\Delta = (\lambda \mu' \nu'') \neq 0$. Then*

- (a) integers l_1, m_1, n_1 , not all zero, may be substituted for x, y, z ,—say $\xi_1 = \lambda l_1 + \lambda' m_1 + \lambda'' n_1, \dots$,—such that

$$|\xi_1| < \sqrt[3]{|\Delta|}, |\eta_1| < \sqrt[3]{|\Delta|}, |\zeta_1| \leq \sqrt[3]{|\Delta|};$$

- (b) integers l_2, m_2, n_2 , not all zero, may be substituted for x, y, z such that

$$|\xi_2| + |\eta_2| + |\zeta_2| < \sqrt[3]{6|\Delta|}.$$

These results follow immediately from Theorem I', the corresponding surface $\varphi = 1$ being a parallelepiped and an octahedron respectively. By a special analysis of the octahedron Minkowski proved (M_4 , page 40) that the left-hand member under (b) is $\leq \sqrt[3]{\frac{108}{19}} |\Delta|$.

From this result we derive $|\xi_2 \eta_2 \zeta_2| \leq \frac{4}{19} |\Delta|$.

Similar results are obtained when the coefficients of ξ and η are conjugate imaginary ($\eta = \bar{\xi}$) and those of ζ real. Thus we find (B_1 , page 233, from Theorem III)

$$|\xi_3 \bar{\xi}_3 \zeta_3| < \sqrt{\frac{2}{7}} |\Delta|.$$

THEOREM V. Let f be a positive definite quadratic form in n variables and of determinant D . Then integers, not all zero, may be substituted for the variables such that the numerical value of f is

$$[f] \leq \frac{4}{\pi} \left\{ \Gamma \left(1 + \frac{n}{2} \right) \right\}^{2/n} D^{1/n}.$$

The author has obtained the limit

$$\frac{2}{\pi} \left\{ \Gamma \left(1 + \frac{n+2}{2} \right) \right\}^{2/n} D^{1/n}$$

by means of Theorem II (B_1 , page 233). The exact (superior) limit to the lowest value $[f]$ (i. e., the limit actually reached by at least one quadratic form of determinant D) has been determined for $n = 2, 3, 4, 5$ (see B_1 , page 233). For any n , we know from Theorem III above that this exact limit, which we shall denote by $4(\rho/J)^{2/n}$, cannot fall below

$$\frac{1}{\pi} \left\{ 2 \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots \right) \Gamma \left(1 + \frac{n}{2} \right) \right\}^{2/n} D^{1/n};$$

i. e.,

$$(1) \quad \frac{1 + 1/2^n + 1/3^n + \dots}{2^{n-1}} \leq \rho \leq \frac{n/2 + 1}{2^{n/2}}.$$

Hence, quadratic forms exist for which the corresponding values of ρ satisfy these inequalities. However, no such form has ever been constructed for large values of n .

The problem of the closest *regular* (latticed) packing of spheres in R_n is equivalent to the problem of finding that positive definite quadratic form in n variables and of given determinant, say $D = 1$, whose least value, other than zero, is the highest possible for the given n and D (M₄, page 74 ff.). The ratio of the space occupied by spheres packed in regular layers in a large cube, say, to the volume of the whole cube, is indeed the number ρ defined above.

The author has recently proved, by an essentially different method, that no matter how the spheres be packed in a large volume V , in a "regular" fashion or not, the ratio of the space occupied by the spheres to the whole volume V is

$$< \frac{n/2 + 1}{2^{n/2}}.$$

It may be of interest to note in passing that it follows from the inequalities (1) satisfied by ρ for certain (though unknown) quadratic forms, that the shot-pile packing of spheres, though the closest packing in space of two dimensions and presumably also in space of three dimensions, is very far from being the closest packing in space of a large number of dimensions (M₄, page 95).

APPLICATIONS OF THE GEOMETRY OF NUMBERS TO ALGEBRAIC NUMBERS.

BY PROFESSOR L. E. DICKSON.

(Read at the Chicago Symposium of the American Mathematical Society,
March 28, 1919.)

1. THE geometry of numbers not only furnishes a concrete geometric image of certain fundamental theorems on algebraic numbers, but also provides a new and attractive method of proving important theorems on algebraic fields. For the sake of concreteness we shall restrict attention to the typical case of the cubic field $F(\theta)$, which is composed of the numbers

$r = X + Y\theta + Z\theta^2$, where X, Y, Z are rational numbers, and θ is a root of a cubic equation with integral coefficients, that of θ^3 being unity. A root of any such equation is called an integral algebraic number.

2. *Basis of a Field*.—While the above number r is an integral algebraic number if X, Y, Z are integers, the converse need not be true. However, there exist integral algebraic numbers $\omega_1, \omega_2, \omega_3$ of the field $F(\theta)$ such that the linear functions of $\omega_1, \omega_2, \omega_3$ with integral coefficients give all the integral algebraic numbers of the field. The essential idea of the geometric proof consists in showing that the points (X, Y, Z) which correspond to the integral algebraic numbers r of our field form a lattice determined by the fundamental parallelopiped whose edges OA, OB, OC are defined as follows: Of the points on the positive X -axis which correspond to integral algebraic numbers of $F(\theta)$, let A be the one nearest the origin O and distinct from O . Of the points in the XY -plane having $Y > 0$ and corresponding to integral algebraic numbers of $F(\theta)$, let B be a point lying as near the X -axis as possible. Finally, of the points with $Z > 0$ and corresponding to integral algebraic numbers of our field, let C be one as near as possible to the XY -plane.

3. *Basis of an Ideal*.—A set of integral algebraic numbers of a field is called an ideal if the sum and difference of any two equal or distinct numbers of the set are numbers of the set and if the product of any number of the set by any integral algebraic number of the field equals a number of the set. Consider the lattice whose points correspond to the integral algebraic numbers $x\omega_1 + y\omega_2 + z\omega_3$ of our field, x, y, z being integers. The points of this lattice which correspond to the numbers of any ideal form a new lattice with a fundamental parallelopiped determined by OA_1, OA_2, OA_3 . If α_i is the number corresponding to the point A_i , the numbers of the ideal are the linear combinations, with integral coefficients, of $\alpha_1, \alpha_2, \alpha_3$, which therefore form a basis of the ideal.

4. *Units of a Field*.—By a unit ϵ is meant an integral algebraic number which divides unity, so that $1/\epsilon$ equals an integral algebraic number. Dirichlet* stated, but did not publish a detailed proof of, the following fundamental theorem: If among the fields conjugate to a field F of degree m , r are real and the remaining $m - 2r = 2c$ form c pairs of conjugate

* *Sitzungsber. Akad. Wiss. Berlin*, 1846, pp. 103-7; *Werke*, I, p. 639 (cf. p. 622, p. 630).

imaginary fields, there exist $s = r + c - 1$ units $\epsilon_1, \dots, \epsilon_s$ in F such that every unit ϵ of F can be expressed in one and but one way in the form

$$\epsilon = \rho \epsilon_1^{a_1} \dots \epsilon_s^{a_s},$$

where $a_1, \dots, a_s$ are rational integers, positive, negative or zero, while ρ is a root of unity belonging to the field F . Kronecker* gave a proof along the lines suggested by Dirichlet. Another algebraic proof had been given by Dedekind.†

Minkowski‡ gave a new proof which is largely algebraic, but makes some use of his geometric results. Later, he gave§ a more purely geometric form to his proof, confining his discussion to the typical cases of cubic and quartic fields. This proof makes use of various ideas and results of the geometry of numbers and shows the power and attractiveness of the latter subject.

5. *Classes of Ideals*.—In his proof of the finiteness of the number of classes of ideals in a cubic field, Minkowski|| made use of his results on the upper bound of a product of three linear forms. The process enables us actually to find representatives of the various existing classes of ideals. This problem would be simplified by the discovery of more exact upper bounds.

PRODUCTS OF SKEW-SYMMETRIC MATRICES.

BY PROFESSOR A. A. BENNETT.

IN the March (1919) number of the BULLETIN occurs (page 281) the following: The philosophical faculty of the University of Berlin announces the following prize problem: "To determine by means of the theory of elementary divisors, the criteria that a given matrix be capable of representation as the composition of two skew-symmetric matrices." A dis-

* *Comptes Rendus*, Paris, vol. 96, 1883; vol. 99, 1884; Werke, III, pp. 1-30.

† Dirichlet-Dedekind, *Zahlentheorie*, ed. 2, 1871, § 166, pp. 471-9; ed. 3, 1879, § 177, pp. 555-567; ed. 4, 1894, § 183, pp. 590-603. Cf. Hilbert, *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 4, 1894-5, pp. 214-222.

‡ *Geometrie der Zahlen*, 1896, pp. 135-147.

§ *Diophantische Approximationen*, 1907, pp. 133-148.

|| *Diophantische Approximationen*, pp. 162-167.

cussion of this topic appears therefore of timely interest. To avoid prolixity, only the case of non-singular matrices will be here considered.

For the sake of brevity and clarity the order of the proof will not be interrupted to prove elementary lemmas which in themselves present no difficulty, but several of such lemmas will be stated without proof at the start. Some of these are proved in all discussions of elementary divisors and others while not so familiar present no new difficulties. The definitions and notations employed are those of Bôcher: *Introduction to Higher Algebra*.

Some Preliminary Lemmas and Definitions.

1. If P is any non-singular matrix, there exist uniquely defined non-singular matrices P' and P^{-1} , known respectively as the conjugate or transposed, and as the inverse of the given matrix. Further, $(P')^{-1} = (P^{-1})'$ and $P^{-1}P = PP^{-1} = I$, where I is the identity, i. e., $MI = IM = M$ for every square matrix M of the same order as I .

2. Skew-symmetric matrices of odd order are necessarily singular, but non-singular skew-symmetric matrices exist of all even orders.

3. If A and B be square matrices of the same order n and with constant elements, the matrix $A + \lambda B$, where λ is variable, is called a linear λ -matrix. The greatest common divisor of all determinants of order j , formed by suppressing $n - j$ rows and $n - j$ columns of $A + \lambda B$, will be a polynomial in λ . If the arbitrary numerical multiplier in the definition of the greatest common divisor be so taken that the non-vanishing term of highest degree in λ has unity for coefficient, this polynomial is denoted by $D_j(\lambda)$. The ratio $D_i(\lambda)/D_{i-1}(\lambda)$, where $D_0(\lambda)$ is arbitrarily defined as unity, is called the i th invariant factor, and is denoted by $E_i(\lambda)$. For i odd, E_i is called an odd invariant factor, and for i even, an even invariant factor.

4. Operations on matrices which merely interchange rows or columns, or multiply rows or columns by non-vanishing factors, or replace a given row by the sum of this row and another, or likewise with columns, are called elementary transformations.

5. If each of two linear λ -matrices is carried into the other by means of elementary transformations, these have the same

invariant factors. Hence, in particular, $PMP^{-1} - \lambda I$ and $M - \lambda I$ have the same invariant factors, for any matrix P of the same order, since the existence of P^{-1} implies that P is non-singular.

6. There exists a standard skew-symmetric matrix T of order $2n$, defined by the condition that if t_{ij} be a general element of T , $t_{ij} = 0$, for $j \neq 2n - i$, $t_{i, 2n-i} = -1$, $0 < i \leq n$, $t_{i, 2n-i} = +1$, $n < i \leq 2n$. This standard matrix T is called "standard," for the reason that for any non-singular skew-symmetric matrix S of order $2n$, there is a non-singular matrix, P , such that $PSP' = T$.

7. If a matrix K of order $2n$ be such as to have throughout among its elements $a_{ij} = a_{n+i, n+j}$, $i \leq n$, $j \leq n$, and $a_{ij} = 0$, if $i \leq n < j$ or $j \leq n < i$, then the submatrix K_0 consisting of the elements a_{ij} , where $i \leq n$, $j \leq n$, determines the invariant factors of K . Indeed if $F_1, F_2, \dots, F_n$, be the invariant factors of K_0 , $E_{2i-1} = E_{2i} = F_i$, $i = 1, 2, \dots, n$, where E_j is the j th invariant factor of K , $j = 1, 2, \dots, 2n$.

8. If S_1 and S_2 are any two non-singular skew-symmetric matrices of the same order, $2n$, the linear λ -matrix $S_1 + \lambda S_2$ may be reduced by elementary transformations to a form $L + \lambda T$, where T is the standard matrix defined above, and the elements a of L satisfy the conditions that $a_{i, 2n-h} = -a_{2n-h, i}$, $i \leq n$, $h \leq n$, $a_{i, 2n-h} = 0$, $i \leq n \leq h$, or $h < n < i$. Further specialization of the skew-symmetric matrix L is also always possible.

Discussion of the Problem.

Let M be the given non-singular matrix for which it is desired to determine whether M is of the form $S_1 S_2$, where S_1, S_2 are skew-symmetric matrices. By (2), M must be of even order, say $2n$, since otherwise both S_1 and S_2 would be singular, and hence M also. One may select a non-singular matrix, P , such that $PMP^{-1} = (PS_1 P')(P'^{-1} S_2 P^{-1})$ is of the form $M_1 = TS$. The matrix $M_1 - \lambda I$, or $TS - \lambda I$, becomes, on changing the sign of the first n rows and rearranging rows, $S + \lambda T$, the S and T having the same meaning as previously. Using (8), this may be put in the form $L + \lambda T$. Changing the sign of the last n rows and again rearranging rows, this may be written $K - \lambda I$, K being as in (7). From this one has the following conclusion:

THEOREM. *A necessary and sufficient condition that a non-singular matrix, M , shall be expressible as the product of two skew-symmetric matrices, viz., $M = S_1 S_2$, is that every even invariant factor of the linear λ -matrix, $M - \lambda I$, shall be equal to the preceding odd invariant factor.**

UNIVERSITY OF TEXAS.

ON THE FIRST FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society April 27, 1918.)

LET l be an odd prime rational integer and consider the cyclotomic field defined by $e^{2i\pi/l}$. A number of questions connected with this field depend on the divisibility of its class number by l and its powers. This class number can be expressed as the product of two integral factors one of which (generally referred to as the first factor) is

$$(1) \quad h = \frac{f(Z)f(Z^3) \cdots f(Z^{l-2})}{(2l)^{\frac{1}{2}(l-3)}},$$

where

$$f(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{l-2} x^{l-2},$$

$Z = e^{2i\pi/l-1}$, r is a primitive root of l , and r_i is the least positive residue of r^i , modulo l .

Kummer† proved that the necessary and sufficient condition that h be divisible by l is that one of the numbers of Bernoulli, B_s , [$s = 1, 2, \dots, (l-3)/2$] is divisible by l , a B being termed divisible by an integer i when its denominator is prime to i and its numerator is divisible by i . Kronecker‡ gave another proof which was reproduced by Hilbert.§

* Otherwise expressed, the condition is that the number of integers within parentheses in the characteristic shall always be even, and these alike in pairs. Thus [(2, 2); (3, 3, 1, 1); (1, 1)] is possible, while [(2, 1); (2, 1)]; [2], and [1, 1] are impossible.

† *Journal für die Mathematik*, vol. 40 (1850.)

‡ *Werke*, vol. 1, p. 93.

§ *Die Theorie der algebraischen Zahlkörper*, Bericht, p. 429.

In the present paper I give an expression for the residue of h modulo l^n , where n is arbitrary, from which it is possible to give necessary and sufficient conditions that h be divisible by a given power of l in terms of Bernoulli numbers. The argument used is a bit different from those employed by Kummer and Kronecker for the special case $n = 1$.

The decomposition of l into ideal prime factors in the field $\Omega(Z)$ shows that one of these prime factors is

$$\mathfrak{P} = (Z - r, l),$$

whence

$$Z \equiv r \pmod{\mathfrak{P}}$$

and

$$(2) \quad Z^{kl^{a'}} \equiv r^{kl^{a'}} \pmod{\mathfrak{P}^{a'+1}}, \quad a' > \frac{l-3}{2},$$

where k is an integer. The right-hand member of (1) is unaltered by the substitution $(Z^{l^{a'}}/Z)$; hence from (1) and (2)

$$(2l)^{\frac{1}{2}(l-3)}h \equiv \prod_i f(r_i^{l^{a'}}) \pmod{\mathfrak{P}^{a'+1}},$$

where $s = 1, 3, \dots, l-2$. Since h and $f(r)$ are rational integers and l is not divisible by $\mathfrak{P}^2$, we then obtain, if $a = a' - \frac{1}{2}(l-3)$

$$(3) \quad h \equiv \frac{\prod_i f(r_i^{l^{a'}})}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^{a+1}}.$$

We also have

$$r_i \equiv r^i \pmod{l}, \quad r_i^{sl^{a'}} \equiv r_i^{sl^{a'}} \pmod{l^{a'+1}}$$

and this gives

$$(3a) \quad \frac{f(r_i^{sl^{a'}})}{l} \equiv \sum_{n=1}^{l-1} \frac{n^{sl^{a'}+1}}{l} \pmod{l^{a'}}.$$

But since

$$n^{l^a(l^{a'}-1)} \equiv 1 \pmod{l^{a+1}},$$

then

$$\sum_{n=1}^{l-1} \frac{n^{sl^{a'}+1}}{l} \equiv \frac{\sum_{n=1}^{l-1} n^{sl^{a'}+1}}{l} \pmod{l^a},$$

and (3) and (3a) give

$$(4) \quad h \equiv \frac{\prod_{s=1}^{l-1} \sum_{n=1}^{s l^a+1} n^{s l^a+1}}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^a}.$$

We shall now show that, for s odd,

$$(5) \quad \sum_{n=1}^{s l^a+1} n^{s l^a+1} \equiv (-1)^{(s l^a-1)/2} l B_{(s l^a+1)/2} \pmod{l^{a+1}}.$$

By the Bernoulli summation formula we have if $b_1 = -1/2$, $b_{2a+1} = 0$, and $b_{2a} = (-1)^{a+1} B_a$, where $B_1 = 1/6$, $B_2 = 1/30$, etc., $b_s = b^s$,

$$(6) \quad \sum_{n=1}^l n^k = l b^k + l \frac{k}{2} l b^{k-1} + l^2 \frac{k(k-1)}{2 \cdot 3} l b^{k-2} \\ + \dots + \frac{l^r}{r+1} \left(\frac{k}{r} \right) l b^{k-r} + \dots$$

By the Staudt-Clausen theorem $l b^a$ is an integer or a fraction whose denominator is prime to l . All the coefficients of $l b^a$ in the right-hand member of (6), commencing with the third, are divisible by l^{a+1} . To prove this, we observe that

$$(b+l)^l \equiv b^l + l^2 b^{l-1} \pmod{l^3}$$

and

$$(b+l)^{s l^a} \equiv b^{s l^a} + s l^{a+1} b^{s l^a-1} \pmod{l^{a+2}},$$

whence

$$(b+l)^{s l^a+2} \equiv b^{s l^a+2} + (s l^a+2) l b^{s l^a+1} + b^{s l^a} l^2 \pmod{l^{a+2}},$$

and

$$(7) \quad \frac{(b+l)^{s l^a+2} - b^{s l^a+2}}{s l^a+2} \equiv l b^{s l^a+1} + \frac{b^{s l^a} l^2}{2} \pmod{l^{a+2}}.$$

The left-hand member of this relation is precisely the right-hand member of (6) if we expand and set $k = s l^a + 1$. Setting b_a for b^a in (7), noting that $b_{s l^a} = 0$ for s odd, and comparing (6) with the new (7), the relation (5) is obtained. We then deduce from (3) and (4)

$$(8) \quad h \equiv \frac{\prod_{s=1}^{l-1} l (-1)^{(s l^a-1)/2} B_{(s l^a+1)/2}}{2^{\frac{1}{2}(l-3)}} \pmod{l^a},$$

$$s = 1, 3, \dots, l-2.$$

Kummer* has shown that

$$\frac{B_a}{a} \equiv (-1)^{k\mu} \frac{B_{a+k\mu}}{a+k\mu} \pmod{l},$$

where k is an integer and a is not a multiple of $\mu = (l-1)/2$. This gives

$$(-1)^{s\mu} \frac{B_{(s+1)/2}}{s(l+1)/2} \equiv \frac{B_{(s+1)/2}}{(s+1)/2} \pmod{l},$$

and applying the latter relation to (8) for $a = 1$, we obtain Kummer's result to the effect that the necessary and sufficient condition that h be divisible by l is that one of the Bernoulli numbers B_s , ($s = 1, 2, \dots, \frac{1}{2}(l-3)$) is divisible by l .

BALA, PA.,
November, 1918.

CORRECTIONS AND NOTE TO THE CAMBRIDGE COLLOQUIUM OF SEPTEMBER, 1916.

BY PROFESSOR G. C. EVANS.

1. *Corrections.* On page 35 in equation (9') change γ to α . In all the formulas and equations following on page 35 change γ to β and β to α .

On page 37 in equation (12) change γ to β and β to α .

On page 39 in the second equation there should be an i as a factor of each of the last two terms of the integrand.

2. *Note to Art. 27, the Analogue of Green's Theorem.* The approach to the analogue of Green's theorem is clearer if made in the following way, and bears more relation to the development with which we are familiar in calculus. The meaning of equations (17) to (20) is perhaps not clear, as the equations stand. But the invariant $H_{\phi_1\phi_1'}$, defined by

$$(21) \quad (V_1 \times V_2)H_{\phi_1\phi_1'} = W_1 \times W_2',$$

which may be rewritten in the new forms

$$\begin{aligned} H_{\phi_1\phi_1'} &= (\beta \cdot W_1)(\beta \cdot W_1') + (\alpha \cdot W_1)(\alpha \cdot W_1') \\ &= -(\beta \cdot W_1)(\alpha \cdot W_2') + (\alpha \cdot W_1)(\beta \cdot W_2'), \end{aligned}$$

* L. c., vol. 41, p. 368.

as we see below, bears a noticeable resemblance to the scalar product of two gradients which is subjected to the familiar analysis of Green's theorem.

The forms just written, however, do not lend themselves to the customary integration by parts. Let us then in these formulas replace the part which refers to, say, W_1 and W_2 by functions φ_1 and φ_2 which depend on them (retaining meanwhile the quantities W_1' and W_2') and seek to write $H_{\alpha_1\alpha_2'}$ in the form

$$H_{\alpha_1\alpha_2'} = W_1' \cdot \nabla \varphi_1 + W_2' \cdot \nabla \varphi_2$$

or more generally, introducing an arbitrary point function $k(x, y, z)$, in the form

$$(22) \quad kH_{\alpha_1\alpha_2'} = W_1' \cdot \nabla \varphi_1 + W_2' \cdot \nabla \varphi_2.$$

Green's theorem is merely the result of integrating both sides of (22) over a three-dimensional region, and reducing the right-hand member by an integration by parts.

What conditions must φ_1, φ_2 satisfy to make possible the equation (22)? A short analysis shows that a necessary and sufficient condition is given by equation (17), which may be deduced directly, starting from this point of view.

The analysis follows. In the first place, by making use of the identities (see (8) page 34)

$$V_1 \alpha \cdot W_1 + V_2 \beta \cdot W_1, \quad W_2' = V_1 \alpha \cdot W_2' + V_2 \beta \cdot W_2',$$

$$V_2' = (V_1 \times V_2) \{ (\alpha \cdot W_1)(\beta \cdot W_2') - (\beta \cdot W_1)(\alpha \cdot W_2') \},$$

from the definition (21),

$$H_{\alpha_1\alpha_2'} = - (\beta \cdot W_1)(\alpha \cdot W_2') + (\alpha \cdot W_1)(\beta \cdot W_2').$$

conditions of isogeneity (10''), we get the other new mentioned for $H_{\alpha_1\alpha_2'}$, as well as additional ones of the character.

equation just written, by comparison with (22), gives equation

$$(\beta \cdot W_2') - (\beta \cdot W_1)(\alpha \cdot W_2') = W_1' \cdot \nabla \varphi_1 + W_2' \cdot \nabla \varphi_2.$$

right-hand member the following substitutions may be

made, to make the two members similar in construction:

$$\begin{aligned} W_1' \cdot \nabla \varphi_1 &= \nabla \varphi_1 \cdot (V_1 \beta - V_2 \alpha) \cdot W_2' \\ &= (V_1 \cdot \nabla \varphi_1)(\beta \cdot W_2') - (V_2 \cdot \nabla \varphi_1)(\alpha \cdot W_2'), \\ W_2' \cdot \nabla \varphi_2 &= \nabla \varphi_2 \cdot (V_1 \alpha + V_2 \beta) \cdot W_2' \\ &= (V_1 \cdot \nabla \varphi_2)(\alpha \cdot W_2') + (V_2 \cdot \nabla \varphi_2)(\beta \cdot W_2'). \end{aligned}$$

Hence the equation becomes

$$\begin{aligned} (\nabla \varphi_1 \cdot V_1 + \nabla \varphi_2 \cdot V_2 - k\alpha \cdot W_1)(\beta \cdot W_2') \\ + (\nabla \varphi_2 \cdot V_1 - \nabla \varphi_1 \cdot V_2 + k\beta \cdot W_1)(\alpha \cdot W_2') = 0. \end{aligned}$$

By hypothesis, φ_1 , φ_2 , k are independent of W_1' , W_2' ; hence by putting $W_2' = V_2$ and V_1 successively the following equations are deduced

$$\begin{aligned} k\alpha \cdot W_1 &= \nabla \varphi_1 \cdot V_1 + \nabla \varphi_2 \cdot V_2, \\ k\beta \cdot W_1 &= \nabla \varphi_1 \cdot V_2 - \nabla \varphi_2 \cdot V_1, \end{aligned}$$

which with the identity $W_1 = V_1 \alpha \cdot W_1 + V_2 \beta \cdot W_1$ yield the result

$$kW_1 = (V_1 V_1 + V_2 V_2) \cdot \nabla \varphi_1 + (V_1 V_2 - V_2 V_1) \cdot \nabla \varphi_2,$$

which is merely (17). Vice versa, from (17) may be deduced (22). Equation (18) is equivalent to (17) by the relation of isogeneity.

In a similar manner equation (19) or (20) is seen to be a necessary and sufficient condition for the functions φ_1' and φ_2' to satisfy in order that $H_{\phi_1 \phi_1'}$ may be written in the form (22').

Equation (17) is equivalent to two independent differential equations on the two functions φ_1 , φ_2 to be determined, as is seen in the equivalence of that equation to the two preceding (i. e., the conditions in the plane $V_1 V_2$). Hence the function k remains entirely arbitrary, and may if we choose be taken everywhere positive.

PARIS, FRANCE,
February, 1919.

CIRCLE AND SPHERE GEOMETRY.

A Treatise on the Circle and the Sphere. By JULIAN LOWELL COOLIDGE. Oxford, the Clarendon Press, 1916. 8vo. 603 pp.

AN era of systematizing and indexing produces naturally a desire for brevity. The excess of this desire is a disease, and its remedy must be treatises written, without diffuseness indeed, but with the generous temper which will find room and give proper exposition to true works of art. The circle and the sphere have furnished material for hundreds of students and writers, but their work, even the most valuable, has been hitherto too scattered for appraisal by the average student. Cyclopedias, while indispensable, are tantalizing. Circles and spheres have now been rescued from a state of dispersion by Dr. Coolidge's comprehensive lectures. Not a list of results, but a well digested account of theories and methods, sufficiently condensed by judicious choice of position and sequence, is what he has given us for leisurely study and enjoyment.

The reader whom the author has in mind might be, well enough, at the outset a college sophomore, or a teacher who has read somewhat beyond the geometry expected for entrance. Beyond the sixth chapter, however, or roughly the middle of the book, he must be ready for persevering study in specialized fields. The part most read is therefore likely to be the first four chapters, and a short survey of these is what we shall venture here.

After three pages of definitions and conventions the author calls "A truce to these preliminaries!" and plunges into inversion. Circles orthogonal to a fixed circle are self-inverse, and this property generalized defines anallagmatic curves. A family of such circles will have an anallagmatic envelope, for example. A novel term, deferrent, means the locus of centers of a family of orthogonal circles. In 35 closely linked theorems we advance to mutually tangent circles, and find at once a demonstration of Steiner's favorite theorem on poristic systems of circles, whose concise formulation warrants quotation (page 34): "Given two non-intersecting circles which possess the property that a ring of n circles may be

constructed all tangent to them and successively tangent to one another making m complete circuits, and if two circles of the ring touch the original ones at points on one circle orthogonal to these two, then the original circles are members of a ring of n_1 circles making m_1 complete circuits, all tangent to the two of the first ring, where

$$\frac{m}{n} + \frac{m_1}{n_1} = \frac{1}{2}.$$

Proving soon after this a condition given by Casey for four mutually external circles tangent to a fifth, the author reduces the four to points upon the fifth circumference and so derives Ptolemy's theorem: "If a convex quadrilateral be inscribed in a circle, the sum of the products of the opposite sides is equal to the product of the diagonals," these lengths being the segments on common tangents employed in Casey's formula. The converses of both are established; then follows the nine-point circle.

A section on circles related to a triangle brings in the names of Euler, Simson, Barrow, Fontené, Nagel, Fuhrmann, Greiner, and Kantor. We cite one theorem whose lowest case is commonly given in textbooks (page 48). "If a polygon be inscribed in a circle and tangents be drawn at all of its vertices, the product of the distances of any point of the circle from these tangents is equal to the product of its distances from the side-lines."

Brocard points of a triangle, Lemoine's circles, Steiner's point, Tarry's point, and others, fill a twenty-page section on the geometry of the triangle (not so entitled, the author restricting his selection severely). Concurrent circles and concyclic points then claim attention,—Miquel's circle, and others due to Grace, Pesci, and our author himself (see *Annals of Mathematics*, series 2, volume 12). Coaxal systems of circles (18 pages) close the first and properly the longest chapter, on the circle in elementary plane geometry. We quote two of the shortest theorems. "The three circles on the diagonals of a complete quadrilateral as diameters are coaxal." This is ascribed to Gauss and Bodenmiller, 1830. "If a circle so moves that it cuts two others in diametrically opposite points, or cuts one in diametrically opposite points and the other orthogonally, it will generate a coaxal system" (page 108).

"Suggestions" are appended to this and to later chapters, such as this: "It seems likely that there are other simple criteria for various systems of tangent circles like Casey's condition for four circles tangent to a fifth, Vahlen's criterion for poristic systems, or the Euler conditions that there may be a triangle or a quadrilateral inscribed in one circle which is circumscribed to the other. . . . It seems likely that there are other chains of concurrent circles and concyclic points besides those noticed in theorems 162-6," etc. Comments of this kind, following protracted expositions reaching often a high degree of intrinsic beauty, are eminently in place.

Chapter 2 treats of the circle in cartesian plane geometry. A labored section on the treatment of circles by trilinear coordinates bears out the conclusion that those coordinates "are not adapted to dealing with the circle in any broad way," but "only in studying those properties of a circle which are related to a particular triangle." Tetracyclic coordinates are therefore introduced, so defined that if (xy) denotes the bilinear sum $\sum_1^4 x_i y_i$, points or zero circles have $(xx) = 0$. A neat form of identical relation among ten circles, ascribed either to Darboux or to Frobenius, is then much employed in specialized applications. It consists in the vanishing of a five-rowed determinant, obtained by separating the ten circles into two sets of five, denoted by x, y, z, s, t and x', y', z', s', t' , each letter indicating of course four tetracyclic coordinates. The constituents of the determinant are then the 25 bilinear sums like

$$(xx'), (xy'), \dots, (tx'), (ty'), \dots, (tt').$$

The examples given, 21 in number, are all decidedly interesting.

Of greater interest is doubtless the fourth section of this chapter, on analytical systems. Some of Morley's memorable work figures here. A less known line of discussion is that on a general cubic series of circles (page 159 seq.). "The common orthogonal circles to corresponding triads in three projective pencils of circles whereof no two have a common member will generate a general cubic series, and every general cubic series may be so generated in ∞ ways." Also: "The locus of the centers of the circles of a general cubic series is a rational curve of the third order." Obviously the subject is the direct homotype of the twisted cubic curve in point space of three dimensions; but as incidence in space is orthogonality between

circles, this version is a possible source of fruitful suggestions. Congruences, or two-parameter families, receive due attention, and the field partly developed is recommended for further exploration.

Famous problems in construction, the third chapter, is unique in the full discussion it gives of each problem with respect to Lemoine's characters of simplicity and exactness. The chief problems are those of Apollonius (to construct a circle tangent to three given circles), of Steiner (to construct a circle meeting three given circles at given angles), Malfatti's (to construct three circles, each of which shall touch the other two, and two sides of a given triangle), and two of Fiedler's. The chapter closes with an account of Mascheroni's geometry of the compasses.

Of later chapters, on more special themes, we mention those on pentaspherical space, on circle transformations and sphere transformations, on the oriented circle (and Laguerre transformations); on algebraic systems of circles in space, and on oriented circles in space. In particular we commend to readers the "suggestions" (page 473) that close the chapter on circle crosses; and the paragraph (page 482) on the pentacycle of Stephanos, and that which follows it.

The book has excellent indices, and an added page of errata and addenda. As a whole, it is a welcome gift to American and English geometers. It has in prospect a long career of usefulness as a compendium of accessible fact and as a source of stimulation and suggestion. Incidentally it affords the refreshing detached humor that marks it as authentic.

H. S. WHITE.

SHORTER NOTICES.

Unified Mathematics. By L. C. KARPINSKI, H. Y. BENEDICT and J. W. CALHOUN. New York, D. C. Heath and Company, 1918. viii + 522 pp.

THE increasing popularity of a unified course in mathematics for college freshmen is reflected in the large number of text books covering this field which have recently been published. Moreover, the movement towards the general adoption of

such a course will undoubtedly be much accelerated by the variety of books available. This movement is clearly a logical one and in harmony with the general tendency in educational practice to avoid useless repetition and to emphasize the co-ordination of related ideas. A properly unified course should make for a better understanding of the nature of mathematics and a more usable knowledge of the mathematical principles that have been studied.

The work under consideration here has many excellent features. One of its principal merits may be found in the interesting way in which the theories developed are illustrated by numerous applications which are of real importance in the world of to day. This particular characteristic of the book is all the more commendable in view of the fact that many mathematical texts are rather lacking in this respect. As a result, it is quite possible for students who have completed a first course in calculus to have a very inadequate idea of the large number of useful applications of the theories they have learned. This ought not to be the case; there is just as good reason for discussing the applications in a course on mathematical theory as in a course on chemical theory.

It is the firm conviction of the reviewer that teachers of mathematics can do much toward fortifying the position of their subject in general education by writing and using texts that lay stress on the modern applications of mathematical theory. In this way courses in mathematics will become richer in content and more interesting for the average student, and the critics of mathematics will be largely disarmed. "Unified Mathematics" furnishes an excellent illustration of what can be done in this direction by writers who are willing to make the effort.

We will now consider the book somewhat more in detail. It may be stated in the first place that the subject matter includes all the more important topics of algebra, trigonometry and analytical geometry that are ordinarily treated in freshman courses. Moreover, excellent use has been made of the relationship between these three fields and a real unification has been achieved. The arrangement of material is good both from a logical and a pedagogical standpoint. Throughout the book a proper emphasis is laid on computation and one of the earlier chapters is devoted to showing how the application of certain algebraic formulas to arithmetical com-

putations serves to diminish considerably the labor of the computer. The introduction of sets of "timing exercises" in cases where the problems to be solved are of a rather mechanical nature, is a novel feature and ought to help in developing greater facility in mathematical technique on the part of the student.

As stated above, the book is unusually rich in applications. A list of the more important and novel ones will readily illustrate this fact. A treatment of compound interest is given in connection with the chapter on exponents and logarithms. A discussion of annuities, bonds at premium and discount, and sinking funds is found in the chapter on geometrical progressions. Railroad curves are treated in the chapter on right triangles and the laws for reflection and refraction of light are stated and applied in connection with the chapter on oblique triangles. In the chapters on the ellipse and parabola the uses of these curves in architecture is explained and problems connected with elliptical and parabolic arches are introduced. In a special chapter on applications of conic sections we find discussions of Kepler's laws, projectiles, reflectors, and elliptical gears. In the chapter on wave motion there is a brief discussion of sound waves, light waves, and electrical waves; a treatment of piston rod motion is also included. In a chapter entitled "Laws of Growth" various applications of the exponential function in connection with physical and biological phenomena are treated. Among the topics discussed are the law of organic growth, Halley's law for the decrease in air pressure, damped vibrations, and the curve of healing of a wound.

A student that has been made acquainted with such a wide range of applications for the theories he has learned will certainly have a great deal more respect for the usefulness of mathematical knowledge. Moreover, his interest in the subject will undoubtedly be stimulated and he will be more apt to continue his work in mathematics after the freshman year. As such results are undoubtedly desired by all serious minded teachers of mathematics, "Unified Mathematics" ought to be widely used.

CHARLES N. MOORE.

Vorlesungen über Zahlen- und Funktionenlehre. Erster Band, erste Abteilung. By ALFRED PRINGSHEIM. Leipzig, B. G. Teubner, 1916.

THIS is the first part of the first volume of an introductory treatise on the theory of functions of real and complex variables. The author proposes to include in the first volume, in addition to a detailed logical treatment of the rational numbers, a complete exposition of the material treated by him in his two articles in the *Encyklopädie*: "Irrationalzahlen und Konvergenz unendlicher Prozesse" and "Unendliche Prozesse mit komplexen Termen," the only part omitted being the theory of infinite determinants. The part of the first volume under review concerns itself with the following principal topics: the system of real numbers, convergent sequences of such numbers, limits, orders of infinity and of infinitesimals, and double sequences of real numbers.

The book is intended primarily for students who are beginning the study of higher mathematics, and the theory is built up throughout from first principles. In fact the author declares in his preface that no previous mathematical knowledge is in general necessary for the comprehension of the first volume. In spite of the elementary character of the exposition, rigor of proof is nowhere sacrificed and the reader is made acquainted with the modern viewpoints in the domain of analysis that is treated.

CHARLES N. MOORE.

I Numeri reali e l'Equazione esponenziale $a^x = b$ per le Scuole Medie Superiori. By Dr. GAETANO FAZZARI. Palermo, Libreria Scientifica D. Capozzi, 1918. 75 pp. Lire 1.80.

THIS little book, as its title indicates, is devoted mainly to the theory of real numbers. It has no scientific pretense, as the author states in the short preface, but contains an exposé of the theory as Dr. Fazzari has presented it for several years to his students at the R. Liceo Umberto I in Palermo.

The book is divided into two parts. In part I is found the theory of real numbers, the development being essentially that of Dedekind. It commences with the definition of a class of rational numbers; i. e., a series of rational numbers satisfying a given condition. Of two classes A and A' , A is said to be inferior to A' , and A' superior to A , if each element of A is less than every element of A' .

There follows the notion of contiguity. Two classes are contiguous if one is inferior to the other and if it is possible to find an element of one and an element of the other such that their difference is less than any given rational number, not zero, arbitrarily small. This definition is indicated symbolically by writing (A, A') with the conditions $a' > a$ and $a_n' - a_m < \epsilon$.

A number α is the limit of two contiguous classes (A, A') if $a \leq \alpha \leq a'$, and this is denoted by writing $\alpha = (A, A')$.

There follows the existence theorem: There exist contiguous classes of rational numbers which do not have a rational limit. The proof is essentially that of Dedekind. The converse is a very simple statement of the Dedekind postulate: If two contiguous classes do not have a rational limit, we shall say that there is one number, and but one, called an irrational number, which is their limit.

The class of all rational numbers and all irrational numbers constitute the class of real numbers.

Sections 2 to 9 inclusive of part I deal with the conditions for the equivalence of two real numbers $\alpha = (A, A')$ and $\beta = (B, B')$, and with the arithmetic operations of addition, subtraction, multiplication, division, involution, and evolution as applied to real numbers defined by contiguous classes.

The first two sections of part II deal with the theory of indices, the exponent being any real number. The last section is devoted to the solution of the exponential equation $a^x = b$.

A series of exercises is inserted after each section, dealing with the subject matter of that section, and there are numerous examples throughout the book to illustrate definitions and theorems.

The treatment is remarkably clear, but one would expect, even in an elementary treatise, some reference to the sources from which the material is drawn. No mention is made, however, of Dedekind or of others who have worked in this field.

Students of mathematics in this country would find no especial difficulty in mastering the contents of Dr. Fazzari's book, but would be unappreciative of the necessity or the beauty of the theory, at least until they were quite beyond the elementary mathematics of our secondary schools.

L. WAYLAND DOWLING.

Infinitesimal Calculus. By F. S. CAREY. Longmans, Green and Company, 1917. Section 1, 144 pp.; Section 2, 207 pp.

THIS text is bound in two separate sections, the first containing sufficient material for a good elementary course, while the two sections together cover the topics usually presented in a longer elementary course. Chapter one reminds one of function theory, for the author treats of such topics as sequence of numbers, the arithmetical continuum, closed and open ranges, etc. The reviewer doubts if a beginner can grasp such concepts. Chapter two, on "Limits," contains many interesting examples to illustrate what a limit is, but nowhere is there to be found a concise definition of the word. The question of left hand and right hand limit, the question of the limit of a sum, product and quotient of two functions, is very thoroughly discussed. In the third chapter the rules for the derivative of a sum, a product and quotient of two functions are derived, but the last two derivations are very blind. The next chapter "The sign of the differential coefficient" treats of maxima and minima, and many fine examples are to be found among the exercises. This is followed by a chapter on algebraic functions. The remaining topics treated in the short elementary course are: "The inverse of a function," "Function of a function," "Tangent and normal," "Parametric equations," "Point of inflexion," "Circle of curvature," "Order of magnitude," "Inverse differentiation," "Logarithmic functions," "Areas," "Volume," "Parabolic approximation," "Simpson's rule," "Moments" and "Center of gravity." No definite integrals are used, in fact the symbol $\int$ is not introduced.

Section two starts with an excellent chapter on exponential and hyperbolic functions. The results of several integration formulas are expressed in terms of these functions. This is followed by a discussion of the motion of a particle along an axis. The definite integral is now introduced and many of the elementary properties which we usually assume in an elementary course are proved in full detail. This is followed by a chapter on polar coordinates in which pedal curves and intrinsic equations are discussed together with the usual material to be found in such a chapter. Work on partial differentiation, double integration, triple integration, expansion in power series, curve tracing, singular points, Newton's method of ascertaining the form of a curve at the origin and

at infinity, envelopes, involutes, roulettes and planimeters, finish the work in the calculus. This is followed by a short course in differential equations and a short but very interesting chapter on nomography.

The book is well written and the typographical errors are few, there being more, however, in the second section than in the first.

F. M. MORGAN.

Annuaire du Bureau des Longitudes pour l'An 1919. Paris, Gauthier-Villars.

THERE are few changes in the *Annuaire* for 1919 which call for special notice. Every year the editors have evidently to consider the pressure of matter for which there is a demand and the chief problem appears to be in finding out what can be omitted in order to make room for new data. The tendency seems to be in the direction of making the volume more useful to the person with scientific interests, rather than to the general reader, for we notice that articles on such subjects as coinage, legal measures, and geography are either suppressed or cut down.

An interesting summary of our present knowledge of the figures of the equilibrium of a liquid is furnished by M. Appell. The account is free from mathematical developments and it is written in a style which gives the main facts; there is an excellent list of the chief memoirs on the subject attached to the end of the article. This subject has been lately revived on account of the papers continuing Darwin's work in the *Transactions of the Royal Society* by Jeans. M. Hamy, in a second appendix, discusses the possibilities of the determination of the actual dimensions of stars by interference methods. He believes that some indications of value can be obtained in this way in spite of the extremely minute angles that the diameters of even the nearest stars subtend at the earth.

E. W. BROWN.

NOTES.

THE twenty-sixth summer meeting of the American Mathematical Society will be held at the University of Michigan, Ann Arbor, Mich., on Tuesday, Wednesday, and Thursday, September 2-4, 1919, beginning on Tuesday afternoon at two o'clock. The meeting will be followed by that of the Mathematical Association of America on September 4-5. A joint session of the two organizations is planned for Thursday afternoon, and a joint dinner for Thursday evening. Abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by August 12 if it is desired that they should be printed in advance of the meeting; otherwise they may be sent in by August 18.

THE committee on arrangements for the summer meeting and colloquium of the Society to be held at the University of Chicago in 1920 announce the following lecturers and provisional subjects: Professor G. D. BIRKHOFF: "Dynamical systems;" Professor F. R. MOULTON: "Certain topics in functions of infinitely many variables."

AMONG the subjects to be considered at the series of scientific conferences to be held at Brussels, beginning July 18, is the formation of an international mathematical union. Such unions have already been organized in astronomy, chemistry, and geophysics. Whatever tentative plans may evolve for a mathematical union will be submitted to the various mathematical organizations of the world for discussion.

THE General education board has appropriated \$16,000 for salaries and other expenses of the National committee on mathematical requirements. Chairman J. W. YOUNG, as representative of the colleges, and vice-chairman J. A. FOBERG, as representative of the secondary schools, will devote their whole time during the coming year to the work of the committee, which will cover the revision of secondary school and college courses and of college entrance requirements in mathematics.

THE technical staff of the U. S. ordnance department has been authorized to secure the services of five experts in mathematics and dynamics, at salaries ranging from \$2,500 to \$5,000, to conduct scientific research on ordnance problems, act as advisers on all mathematical and scientific problems for the ordnance department, and keep up connections between the department and the scientific world.

At the meeting of the London mathematical society held April 24, the following papers were read: By K. A. RAU, "Lambert's series" and "The relations between the convergence of a series and its summability by Cesàro means"; by G. H. HARDY and J. E. LITTLEWOOD, "Tauberian theorem for Lambert's series"; by W. H. YOUNG, "A formula for an area."

At the meeting of May 15 the following papers were read: By G. N. WATSON, "Zeros of Lommel's polynomials"; by W. H. YOUNG, "The triangulation method of defining the area of a surface."

At the meeting of the Edinburgh mathematical society on June 13 the following papers were read: By W. L. MARR, "Systems of apolar triads in the twisted cubic" and "The co-symmedian system of tetrahedra in a sphere"; by D. M. Y. SOMMERVILLE, "Approximations to the length of an arc"; by D. G. TAYLOR, "A new approximate formula for the complete elliptic integrals $K(k)$, $E(k)$."

THE Association of teachers of mathematics in the middle states and Maryland held a joint meeting with the Association of mathematics teachers of New Jersey at the Central high school, Newark, on May 3. Papers were read by Professor A. L. JONES, Mr. H. E. WEBB, Professor H. B. FINE, and Mr. P. C. H. PAPPS.

THE following university courses in mathematics are announced:

JOHNS HOPKINS UNIVERSITY (academic year 1919-1920).—By Professor FRANK MORLEY: Higher geometry, three hours (first term); Theory of functions, three hours (second term); Seminar, two hours.—By Professor L. S. HULBURT: Projective geometry and higher plane curves, three hours.—By

Professor A. COHEN: Elementary theory of functions, two hours; Differential equations and mechanics, three hours.—By Dr. F. D. MURNAGHAN: Electricity and magnetism, three hours (first term); Dynamics of a rigid body, three hours (second term).

UNIVERSITY OF CALIFORNIA (academic year 1919–1920).—By Professor M. W. HASKELL: Higher plane curves, three hours (first term); Advanced analytic geometry, three hours (second term).—Professor C. A. NOBLE: Functions of a complex variable, three hours; Elementary mathematics for advanced students, three hours (first term).—By Professor D. N. LEHMER: Theory of numbers, three hours (first term); Algebraic surfaces, three hours (second term).—By Professor FLORIAN CAJORI: History of mathematics, two hours; History of physics, two hours (second term); Seminar, two hours; Teaching of mathematics, three hours (first term).—By Professor T. M. PUTNAM: Solid analytic geometry, three hours (first term); Theory of algebraic equations, three hours (second term).—By Professor FRANK IRWIN: Advanced calculus, three hours.—By Professor B. A. BERNSTEIN: Algebra of logic, three hours (first term); Theory of probability, three hours (second term); Logic of mathematics, two hours.—By Professor THOMAS BUCK: Advanced calculus, two hours.—By Professor J. H. McDONALD: Analytic mechanics, three hours; Partial differential equations, three hours (first term).

UNIVERSITY OF STRASBOURG (second semester, 1919).—By Professor ESCLANGON: Astronomy.—By Professor M. FRÉCHET: General analysis; Advanced calculus.—By Professor E. GAU: Partial differential equations of the second order; Elementary calculus.—By Professor VERONNET: Astronomy.—By Professor VILLAT: Aerodynamics; Mechanics.

At the celebration, on September 21, 1918, of the 250th anniversary of the founding of the University of Lund, honorary doctorates were conferred on GUSTAF ENESTRÖM, editor of *Bibliotheca Mathematica*, and on J. L. W. V. JENSEN, engineer and member of the editorial board of *Acta Mathematica*.

THE Royal academy of sciences of Denmark offers its gold medal for a memoir on the following subject: "To lay the

foundations of a general theory of surfaces not everywhere convex, projectively closed, and without double lines. It is desirable to reduce as much as possible the hypotheses relative to the infinitesimal properties of the surfaces." Competing memoirs may be written in Danish, Norwegian, Swedish, English, German, French or Latin, and should be sent to Professor KNUDSEN, of the Polytechnic school of Copenhagen, before October 31, 1919.

THE Ackermann-Teubner prize for 1918 was awarded to Professor L. PRANDTL, of Göttingen, for his memoir "Ueber den Luftwiderstand von Kugeln" in the *Göttinger Nachrichten* for 1914.

DR. W. GROSS, of the University of Vienna, has been promoted to a professorship of mathematics.

PROFESSOR E. COSSERAT, of the University of Toulouse, has been elected non-resident member of the Paris Academy of sciences.

PROFESSOR E. GOURSAT, of the University of Paris, has been elected member of the Paris academy of sciences, to succeed Professor E. PICARD who has been chosen perpetual secretary of the academy.

PROFESSOR E. PASCAL, of the University of Naples, and Professor E. ALMANZI, of the University of Rome, have been elected national members of the Reale Accademia dei Lincei, of Rome. Professor G. FANO, of the University of Turin, has been elected a corresponding member.

PROFESSOR G. N. WATSON, of the University of Birmingham, has been elected fellow of the Royal society of London.

PROFESSOR WILLIAM BURNSIDE, of the Royal naval college, Greenwich, has retired from teaching.

PROFESSOR JACQUES HADAMARD, of the Collège de France, has accepted an invitation from Yale University to be a Silliman lecturer in the spring of 1920.

MAJOR J. L. COOLIDGE, of Harvard University, gave a course of two lectures a week at the University of Paris on "Geometry in the complex domain" during the spring term.

At the University of California, Professor VITO VOLTERRA, of the University of Rome, will deliver a series of six lectures in August or September. Professor FRANK MORLEY, of Johns Hopkins University, will be head of the department of mathematics for the summer session, and will conduct the mathematical seminar.

PROFESSOR O. D. KELLOGG, of the University of Missouri, has been made a fellow of the American Association for the Advancement of Science, and elected vice-president and chairman of section A.

PROFESSOR E. B. VAN VLECK, of the University of Wisconsin, has accepted an invitation as lecturer on mathematics at Harvard University for the second half of the ensuing academic year. Professor Van Vleck will give, besides a course in the calculus, one on the theory of functions of a complex variable and one on the partial differential equations of mathematical physics.

PROFESSOR M. I. PUPIN, of Columbia University, until recently consul general for Serbia to the United States, has been attending the peace conference at Paris.

PROFESSOR G. C. EVANS, who has been in service as scientific attaché to the American Embassy at Rome, has returned to Rice Institute.

PROFESSOR F. R. SHARPE, of Cornell University, has been promoted to a full professorship of mathematics. Mr. V. C. GROVE has been appointed instructor in mathematics.

At the Sheffield scientific school, Yale University, the following instructors in mathematics have been appointed: Mr. HERMAN BETZ, Dr. E. A. T. KIRCHER, Dr. J. M. STETSON, and Mr. EUGENE TAYLOR.

In the mathematics department of the University of Illinois, Dr. J. R. KLINE, of the Sheffield scientific school, has been appointed associate and Dr. C. C. CAMP has been appointed instructor.

DR. C. A. FISCHER, of Columbia University, has been appointed Seabury professor of mathematics and astronomy at Trinity College, Hartford.

AT the University of Colorado assistant professor G. H. LIGHT has been promoted to an associate professorship of mathematics.

MR. C. A. NELSON has been appointed instructor in mathematics at the University of Kansas.

MR. A. D. CAMPBELL, of Cornell University, has been appointed instructor in mathematics in Yale University.

AT Brown University, Captain R. E. GILMAN, of Cornell University, and Captain R. W. BURGESS have been appointed assistant professors of mathematics, and Mr. C. R. ADAMS instructor in mathematics.

PROFESSOR E. G. BILL, who has been serving as director of the military service branch at Ottawa, Canada, will return to Dartmouth College in the fall.

PROFESSOR P. P. BOYD, of the University of Kentucky, has been elected president of the Kentucky academy of science.

PROFESSOR E. D. ROE, Jr., of Syracuse University, has been elected director of the observatory. His position in the department of mathematics remains unchanged.

CAPTAIN A. L. UNDERHILL, of the University of Minnesota, has been appointed commandant at the University of Grenoble, where several hundred American soldiers are studying.

DR. H. C. M. MORSE and Dr. I. A. BARNETT have been appointed Benjamin Peirce instructors in Harvard University.

MR. RAYMOND DOUGLAS of the University of Maine, Dr. J. S. TAYLOR, of the University of California, and Dr. NORBERT WIENER have been appointed instructors in mathematics in the Massachusetts Institute of Technology.

DR. H. L. SLOBIN, of the University of Minnesota, has been appointed head of the department of mathematics in New Hampshire College.

MR. H. M. SHOWMAN, of the Colorado School of Mines, has been appointed assistant professor of mathematics at the Case school of applied science.

PROFESSOR E. E. MOOTS, of Cornell College, Iowa, has been elected head of the department of mathematics and engineering.

MR. P. W. HILL, who has been in service over seas, will resume his instructorship in mathematics at Wabash College.

MR. H. L. SMITH and Mr. WARREN WEAVER have been appointed instructors in mathematics at the University of Wisconsin.

MISS HELEN BARTON, of Salem College, and Miss MARION E. STARK, of Meredith College, have been appointed instructors in mathematics at Wellesley College.

DR. MARION B. WHITE, of the Michigan State normal college, Ypsilanti, has been appointed professor of mathematics at Carleton College.

The death is announced, on February 27, 1919, of Professor A. M. LIAPOUNOFF, of the University of Petrograd.

PROFESSOR M. B. WEINSTEIN died recently at Berlin in his sixty-fifth year.

MR. ROBERT BOWES, of the firm of Bowes and Bowes, Cambridge, publishers of the *Messenger of Mathematics*, died February 9, 1919, in his eighty-fourth year.

PROFESSOR A. B. NELSON, of Central University, Danville, Ky., died in November, 1918.

BOOK CATALOGUES: Cambridge University Press, list of publications, including over 200 titles in mathematics, physics and chemistry; Librairie Paul Ritti, Paris, list of 45 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIGOURDAN (G.). Les premières sociétés savantes de Paris au XVII^e siècle et les origines de l'Académie des Sciences. Paris, Gauthier-Villars, 1918. 4to. 24 pp. Fr. 1.50

CASHMORE (M.). Fermat's last theorem: three proofs by elementary algebra. Revised edition. London, Bell, 1919. 55 pp. 2s. 6d.

DICKSON (L. E.). History of the theory of numbers. Volume 1: Divisibility and primality. Washington, Carnegie Institution, 1919. Sm. 4to. 12 + 486 pp. Paper \$7.50; cloth, \$8.00

MERCOGLIANO (D.). See YOUNG (J. W.).

MORÁVEK (G.). Allgemeine Beweise der Gültigkeit des letzten Fermatschen Satzes über die unbestimmte Gleichung $z^n = x^n + y^n$. Mit Ergänzungen. Prag, G. Morávek, 1917. 8vo. 2 pp.

— Allgemeine Beweise der Gültigkeit des letzten Fermatschen Satzes. Prag, G. Morávek, 1918. 1 p.

RUSSELL (B.). Introduction to mathematical philosophy. London, Allen and Unwin, and New York, Macmillan, 1919. 8vo. \$3.00

SCHLÜTER (H.). Die höhere Mathematik als allgemeinverständliches Rechnungsmittel. Berlin, Meusser, 1917. 8vo. 50 pp.

VILLANI (N.). L'equazione di Fermat $x^n + y^n = z^n$, con dimostrazione generale. Lanciano, G. Carabba, 1918. 8vo. 17 pp. L. 2.00

YOUNG (J. W.). I concetti fondamentali dell'algebra e della geometria. Versione e note di Domenico Mercogliano. (Manuali Pierro.) Napoli, 1919. L. 8.00

II. ELEMENTARY MATHEMATICS.

BOURDON (P. L. M.). Trigonométrie rectiligne et sphérique. Edition revue et annotée par C. Brisse. Nouveau tirage. Paris, Gauthier-Villars, 1918. 8vo. Fr. 5.00

BRISSE (C.). See BOURDON (P. L. M.).

CAMINATI (C.) e CAMINATI (P.). Nuovo manuale italiano logaritmico-trigonometrico con sette o con dieci decimali. Piacenza, V. Porta, 1918. 8vo. 63 pp. L. 3.50

CAMINATI (P.). See CAMINATI (C.).

GÉRARD (L.). See NIEWENGLOWSKI (B.).

GOYEN (P.). Elementary mensuration. Constructive plane geometry and numerical trigonometry. London, Macmillan, 1919. 8 + 169 pp. 3s. 6d.

HARANG (F.). Eléments de trigonométrie suivis d'une instruction sur la règle à calcul. Paris, Dunod et Pinat, 1919. 8vo. 6 + 138 pp. Fr. 7.50

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The following dates have been fixed for the meetings of the Society :

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New York: October 25, 1919

New York: December 30-31, 1919

